

Dimensional reduction for energies with linear growth involving the bending moment

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Abstract

A Γ -convergence analysis is used to perform a 3D-2D dimension reduction of variational problems with linear growth. The adopted scaling gives rise to a nonlinear membrane model which, because of the presence of higher order external loadings inducing a bending moment, may depend on the average in the transverse direction of a Cosserat vector field, as well as on the deformation of the mid-plane. The assumption of linear growth on the energy leads to an asymptotic analysis in the spaces of measures and of functions with bounded variation.

Résumé

Une analyse variationnelle par Γ -convergence est utilisée pour étudier un problème de réduction de dimension 3D-2D pour des énergies à croissance linéaire. La mise à l'échelle donne lieu à un modèle effectif de membrane qui, en vertu de la présence de forces extérieures engendrant un moment fléchissant, dépend de la moyenne dans la direction transverse du vecteur de Cosserat ainsi que de la déformation de la surface moyenne. L'hypothèse de croissance linéaire nécessite une analyse asymptotique dans les espaces de mesures et de fonctions à variation bornée.

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1 Introduction

In solid mechanics, the equilibrium state of a body may be described by an energy minimization problem. When we deal with very thin structures, *i.e.*, structures whose thickness is much smaller than the other dimensions, it is convenient to consider a lower-dimensional model describing the behavior of the minimizing sequences when the thickness goes to zero in the thin direction. The knowledge of these asymptotic models may be useful, for example, in numerical implementation since it gives less cost of time of calculus.

In the seminal paper [19], the authors derived a nonlinear membrane model from three dimensional nonlinear elasticity, for energies having a polynomial growth of order $p > 1$. They computed the Γ -limit in the Sobolev space $W^{1,p}$ of the elastic energy without any convexity condition. A general integral representation result has been later established in [12] where applications to heterogeneous bodies in the transverse direction, homogenization and optimal design problems are given. The case of completely heterogeneous materials has been carried out in [6]. We also refer to [4, 5, 7, 11] for the study of fractured thin films in the space SBV^p of special functions with bounded variation. In [9], a richer model has been proposed introducing higher order surface loadings. It leads to bending moment effects enhanced, in the asymptotic model, through the explicit dependence on the average in the transverse direction of a Cosserat vector field. A generalization to heterogeneous media has been given in [6] and an abstract integral representation result in $W^{1,p}$ (and also SBV^p) has been proved in [5].

In this paper, we seek to derive a two-dimensional nonlinear membrane model from three-dimensional nonlinear elasticity involving a bulk energy with linear growth ($p = 1$). As in [5, 6, 9] we allow the presence

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of higher order surface loadings inducing a bending moment. Due to the linear growth of the energy, the limit model depends on a two-dimensional deformation which belongs to the space BV of functions with bounded variation, and on a Cosserat vector which is a Radon measure. Note that dimensional reduction problems for energies with linear growth have also been studied in [11] for cracked thin films. In this case, the 3D-energy which is the sum of a bulk and a surface term penalizing the presence of the cracks, is defined in the space SBV .

Let us consider ω a bounded open subset of \mathbb{R}^2 with Lipschitz boundary and set $\Omega_\varepsilon := \omega \times (-\varepsilon/2, \varepsilon/2)$. We assume that Ω_ε stands for the reference configuration of a homogeneous nonlinear elastic thin film whose stored energy density is given by the Borel function $W : \mathbb{R}^{3 \times 3} \rightarrow [0, +\infty)$. Our first main assumption is that W satisfies some linear growth and coercivity conditions, *i.e.*, there exist $0 < \beta' \leq \beta < +\infty$ such that

$$\beta'|\xi| \leq W(\xi) \leq \beta(1 + |\xi|) \quad \text{for every } \xi \in \mathbb{R}^{3 \times 3}.$$

To fix ideas, suppose that the body is clamped on the lateral boundary $\Gamma_\varepsilon := \partial\omega \times (-\varepsilon/2, \varepsilon/2)$, and that the sections $\Sigma_\varepsilon := \omega \times \{\pm\varepsilon/2\}$ are subjected to ε -dependent external loadings $g(\varepsilon) : \Sigma_\varepsilon \rightarrow \mathbb{R}^3$. Assume further that the material is submitted to the action of a body load $f(\varepsilon) : \Omega_\varepsilon \rightarrow \mathbb{R}^3$ so that the total energy of the system, which is given by the difference between the elastic energy and the work of external forces, is

$$\mathcal{E}(\varepsilon)(v) := \int_{\Omega_\varepsilon} W(\nabla v) dx - \int_{\Omega_\varepsilon} f(\varepsilon) \cdot v dx - \int_{\Sigma_\varepsilon} g(\varepsilon) \cdot v d\mathcal{H}^2,$$

for any kinematically admissible deformation field $v : \Omega_\varepsilon \rightarrow \mathbb{R}^3$ satisfying $v(x) = x$ on Γ_ε .

Thanks to the growth condition satisfied by W , we have – at this stage – a good functional setting if we assume any kinematically admissible deformation fields to belong to the space $\mathcal{V}(\varepsilon) := \{\varphi \in W^{1,1}(\Omega_\varepsilon; \mathbb{R}^3) : T\varphi = x \text{ on } \Gamma_\varepsilon\}$, where $T\varphi$ denotes the trace of φ on the lateral boundary Γ_ε . The problem consists in finding equilibrium states of this body, in other words finding minimizers of the functional $\mathcal{E}(\varepsilon)$ over the space $\mathcal{V}(\varepsilon)$.

As explained before, a natural question which arises is the study of the asymptotic behavior of such energies as well as their (eventual) minimizers as the thickness parameter ε tends to zero. This will be performed by means of a Γ -convergence analysis (see *e.g.* [10, 13] for a comprehensive treatment). It is now usual to rescale the problem on a fixed domain $\Omega := \omega \times I$ of unit thickness, where $I := (-1/2, 1/2)$. Similarly set $\Sigma := \omega \times \{\pm 1/2\}$ and $\Gamma := \partial\omega \times I$. Denoting by $x_\alpha := (x_1, x_2)$ the in-plane variable, we define $g_\varepsilon(x_\alpha, x_3) := g(\varepsilon)(x_\alpha, \varepsilon x_3)$, $f_\varepsilon(x_\alpha, x_3) := f(\varepsilon)(x_\alpha, \varepsilon x_3)$, $u(x_\alpha, x_3) := v(x_\alpha, \varepsilon x_3)$ and $\mathcal{E}_\varepsilon(u) = \mathcal{E}(\varepsilon)(v)/\varepsilon$ so that

$$\mathcal{E}_\varepsilon(u) = \int_{\Omega} W\left(\nabla_\alpha u \Big| \frac{1}{\varepsilon} \nabla_3 u\right) dx - \int_{\Omega} f_\varepsilon \cdot u dx - \int_{\Sigma} g_\varepsilon \cdot u d\mathcal{H}^2.$$

Note that since we divided the total energy by ε , we expect to get a term of order ε in the limit model which corresponds, according to the formal asymptotic expansion performed in [17], to a membrane energy which only accounts for stretching effects.

Provided the rescaled external forces f_ε and g_ε have an appropriate order of magnitude (which will be discussed later), it follows from the growth condition satisfied by W and some Poincaré type inequality, that minimizing sequences $\{u_\varepsilon\}$ with finite total energy will be bounded in $W^{1,1}(\Omega; \mathbb{R}^3)$. Actually, the “scaled” gradient of u_ε , *i.e.*, $\{(\nabla_\alpha u_\varepsilon | (1/\varepsilon) \nabla_3 u_\varepsilon)\}$, will be uniformly bounded in $L^1(\Omega; \mathbb{R}^{3 \times 3})$. However, because of the lack of reflexivity of $W^{1,1}(\Omega; \mathbb{R}^3)$, such minimizing sequences will only be relatively compact in the larger space $BV(\Omega; \mathbb{R}^3)$ of functions with bounded variation. Denoting by u any weak* limit in $BV(\Omega; \mathbb{R}^3)$ of the sequence $\{u_\varepsilon\}$, it turns out that the only interesting deformations (according to this scaling) will necessarily satisfy $D_3 u = 0$ in the sense of distributions. Hence u (can be identified to a function which) belongs to $BV(\omega; \mathbb{R}^3)$ and we expect a (Γ) -limit model depending on such deformations.

Our second main assumption is that the (rescaled) surface load can be written as $g_\varepsilon = g_0/\varepsilon + g_1$. It follows from [17, Remark 2.3.2] that, denoting by g_i^\pm ($i = 0$ or 1) the trace of g_i on $\omega \times \{\pm 1/2\}$, the condition $g_0^+ + g_0^- = 0$ must hold. The physical interpretation of this property is that a plate of thickness ε cannot support a non vanishing resultant surface load as $\varepsilon \rightarrow 0$. Assume also for simplicity that $f_\varepsilon = f$. If $\{u_\varepsilon\} \subset W^{1,1}(\Omega; \mathbb{R}^3)$ is a minimizing sequence as above, the work of external forces has the following form

$$\begin{aligned} \mathcal{F}_\varepsilon(u_\varepsilon) &:= \int_{\Omega} f \cdot u_\varepsilon dx + \int_{\Sigma} g_1 \cdot u_\varepsilon d\mathcal{H}^2 + \int_{\omega} g_0^+ \cdot \left(\frac{u_\varepsilon(\cdot, +1/2) - u_\varepsilon(\cdot, -1/2)}{\varepsilon} \right) dx_\alpha \\ &= \int_{\Omega} f \cdot u_\varepsilon dx + \int_{\Sigma} g_1 \cdot u_\varepsilon d\mathcal{H}^2 + \int_{\omega} g_0^+ \cdot \left(\frac{1}{\varepsilon} \int_I \nabla_3 u_\varepsilon(\cdot, y_3) dy_3 \right) dx_\alpha. \end{aligned}$$

Let $u \in BV(\omega; \mathbb{R}^3)$ be an accumulation point of $\{u_\varepsilon\}$ and $\bar{b} \in \mathcal{M}(\omega; \mathbb{R}^3)$ be a weak* limit in the space of Radon measures of the sequence

$$\left\{ \frac{1}{\varepsilon} \int_I \nabla_3 u_\varepsilon(\cdot, y_3) dy_3 \right\}$$

which does always exist up to a subsequence. Taking the limit as $\varepsilon \rightarrow 0$ in the work of external forces, and denoting $\bar{f}(x_\alpha) := \int_I f(x_\alpha, x_3) dx_3$ yields

$$\mathcal{F}_\varepsilon(u_\varepsilon) \rightarrow \mathcal{F}(u, \bar{b}) := \int_\omega (\bar{f} + g_1^+ + g_1^-) \cdot u dx_\alpha + \int_\omega g_0^+ \cdot d\bar{b},$$

provided f , g_1 and g_0 are regular enough, *e.g.*, $f \in L^\infty(\Omega; \mathbb{R}^3)$, $g_1^\pm \in L^\infty(\omega; \mathbb{R}^3)$ and $g_0^+ \in \mathcal{C}_0(\omega; \mathbb{R}^3)$. The presence of this higher order surface load implies the apparition in the limit of the average in the transverse direction of the Cosserat measure \bar{b} which stands for bending moment effects (see [5, 6, 9]). Hence we seek a richer Γ -limit depending on both u and \bar{b} . Note that in general, u and \bar{b} are completely independent macroscopic entities, and as a matter of fact, it may happen that the measures $D_\alpha u$ and \bar{b} are mutually singular (see Example 4.1).

The following theorem is the main result of this work and it describes the behavior of the elastic energy as $\varepsilon \rightarrow 0$. We refer to section 2 for the notations used in the statement.

Theorem 1.1. *Let $\omega \subset \mathbb{R}^2$ be a bounded open set and $W : \mathbb{R}^{3 \times 3} \rightarrow [0, +\infty)$ be a Borel function satisfying (H_1) there exist $0 < \beta' \leq \beta < +\infty$ such that*

$$\beta' |\xi| \leq W(\xi) \leq \beta(1 + |\xi|) \quad \text{for all } \xi \in \mathbb{R}^{3 \times 3};$$

(H₂) there exist $C > 0$ and $r \in (0, 1)$ such that

$$|W^\infty(\xi) - W(\xi)| \leq C(1 + |\xi|^{1-r}) \quad \text{for all } \xi \in \mathbb{R}^{3 \times 3},$$

where W^∞ is the recession function of W .

Then, for every $(u, \bar{b}) \in BV(\Omega; \mathbb{R}^3) \times \mathcal{M}(\omega; \mathbb{R}^3)$, the sequence of functionals

$$J_\varepsilon(u, \bar{b}) := \begin{cases} \int_\Omega W \left(\nabla_\alpha u \middle| \frac{1}{\varepsilon} \nabla_3 u \right) dx & \text{if } \begin{cases} u \in W^{1,1}(\Omega; \mathbb{R}^3), \\ \bar{b} = \frac{1}{\varepsilon} \int_I \nabla_3 u(\cdot, x_3) dx_3, \end{cases} \\ +\infty & \text{otherwise,} \end{cases}$$

Γ -converges for the weak topology of $BV(\Omega; \mathbb{R}^3) \times \mathcal{M}(\omega; \mathbb{R}^3)$ to*

$$E(u, \bar{b}) := \begin{cases} \int_\omega \mathcal{Q}^* W \left(\nabla_\alpha u \middle| \frac{d\bar{b}}{d\mathcal{L}^2} \right) dx_\alpha \\ \quad + \int_{J_u} (\mathcal{Q}^* W)^\infty \left((u^+ - u^-) \otimes \nu_u, \frac{d\bar{b}}{d\mathcal{H}^1 \llcorner J_u} \right) d\mathcal{H}^1 \\ \quad + \int_\omega (\mathcal{Q}^* W)^\infty \left(\frac{dD_\alpha u}{d|D_\alpha^c u|} \middle| \frac{d\bar{b}}{d|D_\alpha^c u|} \right) d|D_\alpha^c u| \\ \quad + \int_\omega (\mathcal{Q}^* W)^\infty \left(0 \middle| \frac{d\bar{b}}{d|\bar{b}^\sigma|} \right) d|\bar{b}^\sigma| \\ +\infty & \text{otherwise,} \end{cases} \quad \text{if } u \in BV(\omega; \mathbb{R}^3), \quad (1.1)$$

where

$$\begin{aligned} \mathcal{Q}^* W(\bar{\xi}|b) &:= \inf_{\lambda, \varphi} \left\{ \int_{Q' \times I} W(\bar{\xi} + \nabla_\alpha \varphi | \lambda \nabla_3 \varphi) dx : \lambda > 0, \varphi \in W^{1,1}(Q' \times I; \mathbb{R}^3), \right. \\ &\quad \left. \varphi(\cdot, x_3) \text{ is } Q' \text{-periodic for } \mathcal{L}^1 \text{-a.e. } x_3 \in I, \lambda \int_{Q' \times I} \nabla_3 \varphi(y) dy = b \right\}, \end{aligned}$$

for all $(\bar{\xi}|b) \in \mathbb{R}^{3 \times 2} \times \mathbb{R}^3$, $(\mathcal{Q}^ W)^\infty$ is the recession function of $\mathcal{Q}^* W$ and \bar{b}^σ is the singular part of \bar{b} with respect to $|D_\alpha u|$ according to the Besicovitch Decomposition Theorem.*

Remark 1.2. The fact that E is the Γ -limit of the family $\{J_\varepsilon\}$ for the weak* topology of $BV(\Omega; \mathbb{R}^3) \times \mathcal{M}(\omega; \mathbb{R}^3)$ means that for every $(u, \bar{b}) \in BV(\omega; \mathbb{R}^3) \times \mathcal{M}(\omega; \mathbb{R}^3)$ and every sequence $\{\varepsilon_j\} \searrow 0^+$, then:

- (i) for any sequence $\{u_j\} \subset W^{1,1}(\Omega; \mathbb{R}^3)$ such that $u_j \xrightarrow{*} u$ in $BV(\Omega; \mathbb{R}^3)$ and $\frac{1}{\varepsilon_j} \int_I \nabla_3 u_j(\cdot, x_3) dx_3 \xrightarrow{*} \bar{b}$ in $\mathcal{M}(\omega; \mathbb{R}^3)$,

$$E(u, \bar{b}) \leq \liminf_{j \rightarrow +\infty} \int_{\Omega} W \left(\nabla_{\alpha} u_j \middle| \frac{1}{\varepsilon_j} \nabla_3 u_j \right) dx;$$

- (ii) there exists a sequence $\{\bar{u}_j\} \subset W^{1,1}(\Omega; \mathbb{R}^3)$ such that $\bar{u}_j \xrightarrow{*} u$ in $BV(\Omega; \mathbb{R}^3)$, $\frac{1}{\varepsilon_j} \int_I \nabla_3 \bar{u}_j(\cdot, x_3) dx_3 \xrightarrow{*} \bar{b}$ in $\mathcal{M}(\omega; \mathbb{R}^3)$, and

$$E(u, \bar{b}) = \lim_{j \rightarrow +\infty} \int_{\Omega} W \left(\nabla_{\alpha} \bar{u}_j \middle| \frac{1}{\varepsilon_j} \nabla_3 \bar{u}_j \right) dx.$$

The strategy used to prove Theorem 1.1 is based on the blow-up method introduced in [14, 15] for the study of the relaxation of integral functionals with linear growth. It rests on a localization of the energy around convenient Lebesgue points, and uses fine properties of measures and BV functions at these points. We adapt here this technique to deal with functionals depending on pairs BV function/measure.

The following result is the analogue of Theorem 1.1 without bending moment. We shall not give a proof of it since it can be deduced from the one of Theorem 1.1 with much easier arguments.

Theorem 1.3. *Let $\omega \subset \mathbb{R}^2$ be a bounded open set and $W : \mathbb{R}^{3 \times 3} \rightarrow [0, +\infty)$ be a Borel function satisfying (H_1) and (H_2) . Then, for every $u \in BV(\Omega; \mathbb{R}^3)$, the sequence of functionals*

$$J_\varepsilon(u) := \begin{cases} \int_{\Omega} W \left(\nabla_{\alpha} u \middle| \frac{1}{\varepsilon} \nabla_3 u \right) dx & \text{if } u \in W^{1,1}(\Omega; \mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases}$$

Γ -converges for the weak* topology of $BV(\Omega; \mathbb{R}^3)$ to

$$E(u) := \begin{cases} \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} u) dx_{\alpha} + \int_{J_u} (\mathcal{Q}W_0)^{\infty}((u^+ - u^-) \otimes \nu_u) d\mathcal{H}^1 \\ \quad + \int_{\omega} (\mathcal{Q}W_0)^{\infty} \left(\frac{dD_{\alpha}^c u}{d|D_{\alpha}^c u|} \right) d|D_{\alpha}^c u| & \text{if } u \in BV(\omega; \mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases}$$

where $W_0(\bar{\xi}) := \inf\{W(\bar{\xi}|b) : b \in \mathbb{R}^3\}$ for all $\bar{\xi} \in \mathbb{R}^{3 \times 2}$, $\mathcal{Q}W_0$ is the 2D-quasiconvexification of W_0 , and $(\mathcal{Q}W_0)^{\infty}$ is the recession function of $\mathcal{Q}W_0$.

The paper is organized as follows: In section 2, we start by introducing some useful notations and basic notions. Then, in section 3 we prove some properties of the different energy densities involved in our analysis. In section 4, we state some properties of the Γ -limit and the last two sections are devoted to the proof of our Γ -convergence result (Theorem 1.1). The lower bound is established in section 5 and the upper bound is proved in the last one.

2 Notations and Preliminaries

Let Ω be a generic open subset of \mathbb{R}^N , we denote by $\mathcal{M}(\Omega)$ the space of all signed Radon measures in Ω with bounded total variation. By the Riesz Representation Theorem, $\mathcal{M}(\Omega)$ can be identified to the dual of the separable space $\mathcal{C}_0(\Omega)$ of continuous functions on Ω vanishing on the boundary $\partial\Omega$. The N -dimensional Lebesgue measure in \mathbb{R}^N is designated as \mathcal{L}^N while \mathcal{H}^{N-1} denotes the $(N-1)$ -dimensional Hausdorff measure. If $\mu \in \mathcal{M}(\Omega)$ and $\lambda \in \mathcal{M}(\Omega)$ is a nonnegative Radon measure, we denote by $\frac{d\mu}{d\lambda}$ the Radon-Nikodým derivative of μ with respect to λ . By a generalization of the Besicovich Differentiation Theorem (see [2, Proposition 2.2]), it can be proved that there exists a Borel set $E \subset \Omega$ such that $\lambda(E) = 0$ and

$$\frac{d\mu}{d\lambda}(x) = \lim_{\rho \rightarrow 0^+} \frac{\mu(x + \rho C)}{\lambda(x + \rho C)}$$

for all $x \in \text{Supp } \mu \setminus E$ and any open convex set C containing the origin.

We say that $u \in L^1(\Omega; \mathbb{R}^d)$ is a function of bounded variation, and we write $u \in BV(\Omega; \mathbb{R}^d)$, if all its first distributional derivatives $D_j u_i$ belong to $\mathcal{M}(\Omega)$ for $1 \leq i \leq d$ and $1 \leq j \leq N$. We refer to [3] for a detailed analysis of BV functions. The matrix-valued measure whose entries are $D_j u_i$ is denoted by Du and $|Du|$ stands for its total variation. By the Lebesgue Decomposition Theorem we can split Du into the sum of two mutually singular measures $D^a u$ and $D^s u$ where $D^a u$ is the absolutely continuous part of Du with respect to the Lebesgue measure \mathcal{L}^N , while $D^s u$ is the singular part of Du with respect to \mathcal{L}^N . By ∇u we denote the Radon-Nikodým derivative of $D^a u$ with respect to the Lebesgue measure so that we can write

$$Du = \nabla u \mathcal{L}^N + D^s u.$$

Let J_u be the jump set of u defined as the set of points $x \in \Omega$ such that there exist $u^\pm(x) \in \mathbb{R}^d$ (with $u^+(x) \neq u^-(x)$) and $\nu_u(x) \in \mathbb{S}^{N-1}$ satisfying

$$\lim_{\rho \rightarrow 0^+} \frac{1}{\rho^N} \int_{\{y \in Q_{\nu_u(x)}(x, \rho) : \pm(y-x) \cdot \nu_u(x) > 0\}} |u(y) - u^\pm(x)| dy = 0,$$

where $Q_\nu(x, \rho)$ denotes any cube of \mathbb{R}^N centered at $x \in \mathbb{R}^N$, with edge length $\rho > 0$, and such that two of its faces are orthogonal to $\nu \in \mathbb{S}^{N-1}$. It is known that J_u is a countably \mathcal{H}^{N-1} -rectifiable Borel set. The measure $D^s u$ can in turn be decomposed into the sum of a jump part and a Cantor part defined by $D^j u := D^s u \llcorner J_u$ and $D^c u := D^s u \llcorner (\Omega \setminus J_u)$. We now recall the decomposition of Du :

$$Du = \nabla u \mathcal{L}^N + (u^+ - u^-) \otimes \nu_u \mathcal{H}^{N-1} \llcorner J_u + D^c u.$$

By Alberti's Rank One Theorem (see [1]), the matrix defined by

$$A(x) := \frac{dD^c u}{d|D^c u|}(x) \in \mathbb{R}^{d \times N}$$

has rank one for $|D^c u|$ -a.e. $x \in \Omega$. If Ω has Lipschitz boundary, we denote by Tu the trace of $u \in BV(\Omega; \mathbb{R}^d)$ (or $u \in W^{1,1}(\Omega; \mathbb{R}^d)$) on $\partial\Omega$.

We now recall basic facts about tangent measures and tangent space to measures referring again to [3] for more details. Let $Q := (-1/2, 1/2)^N$ be the unit cube of \mathbb{R}^N and let $Q(x, \rho) := x + \rho Q$. If $\mu \in \mathcal{M}(\Omega)$ is a non negative Radon measure in Ω and $x \in \Omega$, we denote by $\text{Tan}(\mu, x)$ the set of all non negative finite Radon measures $\nu \in \mathcal{M}(Q)$ such that

$$\frac{1}{\mu(Q(x, \rho_j))} \int_{\mathbb{R}^N} \varphi\left(\frac{y-x}{\rho_j}\right) d\mu(y) \rightarrow \int_Q \varphi(y) d\nu(y),$$

for any $\varphi \in \mathcal{C}_c(\mathbb{R}^N)$ and for some sequence $\{\rho_j\} \searrow 0^+$. The set $\text{Tan}(\mu, x)$ is not empty and for any $t \in (0, 1)$, there exists $\nu \in \text{Tan}(\mu, x)$ such that $\nu(tQ) \geq t^N$ for μ -a.e. $x \in \Omega$ (see [3, Corollary 2.43]).

When $\mu = \mathcal{H}^{N-1} \llcorner S$ for some countably \mathcal{H}^{N-1} -rectifiable set $S \subset \mathbb{R}^N$, we say that S admits an approximate tangent space at $x \in S$ if there exists a $(N-1)$ -dimensional linear subspace π of \mathbb{R}^N such that

$$\frac{1}{\rho^{N-1}} \int_S \varphi\left(\frac{y-x}{\rho}\right) d\mathcal{H}^{N-1}(y) \rightarrow \int_\pi \varphi(y) d\mathcal{H}^{N-1}(y),$$

for any $\varphi \in \mathcal{C}_c(\mathbb{R}^N)$. From [3, Theorem 2.83], we know that \mathcal{H}^{N-1} -a.e. $x \in S$ admits an approximate tangent space. Moreover, the Federer-Volpert Theorem (see [3, Theorem 3.78]) asserts that if $u \in BV(\Omega; \mathbb{R}^d)$, then for \mathcal{H}^{N-1} -a.e. $x \in J_u$, the hyperplane $\nu_u(x)^\perp$ coincides with the approximate tangent space of J_u at x .

In the sequel we will always deal with the cases $N = 2$ or 3 . Let $\omega \subset \mathbb{R}^2$ be a bounded open set and $I := (-1/2, 1/2)$, we define $\Omega := \omega \times I$. We denote by $Q' := (-1/2, 1/2)^2$ the unit cube in \mathbb{R}^2 and if $\nu \in \mathbb{S}^1$, Q'_ν is the unit cube centered at the origin with its faces either parallel or orthogonal to ν . If $x \in \mathbb{R}^2$ and $\rho > 0$, we set $Q'(x, \rho) = x + \rho Q'$ and $Q'_\nu(x, \rho) := x + \rho Q'_\nu$. The canonical basis of \mathbb{R}^2 is denoted by (e_1, e_2) .

Given a matrix $\xi \in \mathbb{R}^{3 \times 3}$, ξ will be written as $(\bar{\xi} | \xi_3)$, where $\bar{\xi} := (\xi_1 | \xi_2) \in \mathbb{R}^{3 \times 2}$ and ξ_i denotes the i -th column of ξ . If $x \in \mathbb{R}^3$, then $x_\alpha := (x_1, x_2) \in \mathbb{R}^2$ is the vector of the first two components of x . The notation ∇_α and ∇_3 denote respectively (approximate) differentiation with respect to the variables x_α and x_3 .

3 Properties of the energy densities

3.1 The bulk energy density

As in [9], we define $\mathcal{Q}^*W : \mathbb{R}^{3 \times 2} \times \mathbb{R}^3 \rightarrow [0, +\infty)$ by

$$\begin{aligned} \mathcal{Q}^*W(\bar{\xi}|b) &:= \inf_{\lambda, \varphi} \left\{ \int_{Q' \times I} W(\bar{\xi} + \nabla_\alpha \varphi | \lambda \nabla_3 \varphi) dx : \lambda > 0, \varphi \in W^{1,1}(Q' \times I; \mathbb{R}^3), \right. \\ &\quad \left. \varphi(\cdot, x_3) \text{ is } Q'\text{-periodic for } \mathcal{L}^1\text{-a.e. } x_3 \in I, \lambda \int_{Q' \times I} \nabla_3 \varphi dy = b \right\}. \end{aligned} \quad (3.1)$$

We recall the main properties of \mathcal{Q}^*W proved in [9, Proposition 1.1].

Proposition 3.1. *Let $W : \mathbb{R}^{3 \times 3} \rightarrow [0, +\infty)$ be a Borel function satisfying (H_1) and let \mathcal{Q}^*W be defined by (3.1). The following properties hold:*

- $CW \leq \mathcal{Q}^*W \leq \mathcal{Q}W$, where CW and $\mathcal{Q}W$ denote, respectively, the convex and quasiconvex envelopes of W ;
- for all $\bar{\xi} \in \mathbb{R}^{3 \times 2}$ and $b \in \mathbb{R}^3$,

$$\beta'(|\bar{\xi}| + |b|) \leq \mathcal{Q}^*W(\bar{\xi}|b) \leq \beta(1 + |\bar{\xi}| + |b|); \quad (3.2)$$

- there holds

$$\mathcal{Q}^*(\mathcal{Q}W) = \mathcal{Q}^*W; \quad (3.3)$$

- let $W_0 : \mathbb{R}^{3 \times 2} \rightarrow [0, +\infty)$ be given by $W_0(\bar{\xi}) := \inf \{W(\bar{\xi}|b) : b \in \mathbb{R}^3\}$ and $\mathcal{Q}W_0$ denotes its 2D-quasiconvex envelope. Then we have

$$\inf_{b \in \mathbb{R}^3} \mathcal{Q}^*W(\bar{\xi}|b) = \mathcal{Q}W_0(\bar{\xi}).$$

We now highlight a convexity property of the energy density \mathcal{Q}^*W .

Proposition 3.2. *The function \mathcal{Q}^*W is convex in the directions $(z \otimes \nu, b)$, with $z, b \in \mathbb{R}^3$ and $\nu \in \mathbb{S}^1$.*

Proof. Let $b_1, b_2 \in \mathbb{R}^3$ and $\bar{\xi}_1, \bar{\xi}_2 \in \mathbb{R}^{3 \times 2}$ be such that $\bar{\xi}_2 - \bar{\xi}_1 = z \otimes \nu$ for some $z \in \mathbb{R}^3$ and $\nu \in \mathbb{S}^1$. Fix also $\theta \in [0, 1]$ and set

$$u(x_\alpha) := \begin{cases} \bar{\xi}_1 x_\alpha + (x_\alpha \cdot \nu)z - (1 - \theta)jz & \text{if } j \in \mathbb{Z} \text{ and } j \leq x_\alpha \cdot \nu < j + \theta, \\ \bar{\xi}_1 x_\alpha + (1 + j)\theta z & \text{if } j \in \mathbb{Z} \text{ and } j + \theta \leq x_\alpha \cdot \nu < j + 1 \end{cases}$$

and

$$A := \{x_\alpha \in \mathbb{R}^2 : \text{there exists } j \in \mathbb{Z} \text{ such that } j \leq x_\alpha \cdot \nu < j + \theta\}.$$

Now define $u_n(x_\alpha) := u(nx_\alpha)/n$ and $\bar{b}_n(x_\alpha) := \chi_A(nx_\alpha) b_2 + (1 - \chi_A(nx_\alpha)) b_1$. Then, by the Riemann-Lebesgue Lemma, $u_n \rightharpoonup (\theta \bar{\xi}_2 + (1 - \theta) \bar{\xi}_1) x_\alpha$ in $W^{1,p}(Q'; \mathbb{R}^3)$ and $\bar{b}_n \rightharpoonup \theta b_2 + (1 - \theta) b_1$ in $L^p(Q'; \mathbb{R}^3)$ for every $p \geq 1$. Using the fact that the functional

$$(u, \bar{b}) \mapsto \int_{Q'} \mathcal{Q}^*W(\nabla_\alpha u | \bar{b}) dx_\alpha$$

is sequentially weakly lower semicontinuous in $W^{1,p}(Q'; \mathbb{R}^3) \times L^p(Q'; \mathbb{R}^3)$ (see e.g. [9, Remark 1.4]), we infer that

$$\begin{aligned} \mathcal{Q}^*W(\theta(\bar{\xi}_2|b_2) + (1 - \theta)(\bar{\xi}_1|b_1)) &\leq \liminf_{n \rightarrow +\infty} \int_{Q'} \mathcal{Q}^*W(\nabla u_n | \bar{b}_n) dx_\alpha \\ &= \lim_{n \rightarrow +\infty} \int_{Q'} [\chi_A(nx_\alpha) \mathcal{Q}^*W(\bar{\xi}_1 + z \otimes \nu | b_2) \\ &\quad + (1 - \chi_A(nx_\alpha)) \mathcal{Q}^*W(\bar{\xi}_1 | b_1)] dx_\alpha \\ &= \theta \mathcal{Q}^*W(\bar{\xi}_2|b_2) + (1 - \theta) \mathcal{Q}^*W(\bar{\xi}_1|b_1), \end{aligned}$$

which is the desired result. \square

We also remark that we could arrive at the same conclusion by observing that the function \mathcal{Q}^*W is \mathcal{A} -quasiconvex (see [16], page 1369, Example (iii)) with respect to the operator $\mathcal{A} := (\text{curl}, 0)$, where

$$\mathcal{A} : (F|\bar{b}) \mapsto (\text{curl}F, 0)$$

with $F : \mathbb{R}^2 \rightarrow \mathbb{R}^{3 \times 2}$ and $b : \mathbb{R}^2 \rightarrow \mathbb{R}^3$. Indeed, by virtue of [16, Proposition 3.4], the function \mathcal{Q}^*W turns out to be convex in the directions $(z \otimes \nu, b)$, with $z, b \in \mathbb{R}^3$ and $\nu \in \mathbb{S}^1$.

The following result asserts that in the definition (3.1) of \mathcal{Q}^*W , one can replace the cube Q' by any rotated cube Q'_ν .

Proposition 3.3. *Let $W : \mathbb{R}^{3 \times 3} \rightarrow [0, +\infty)$ be a Borel function satisfying (H_1) , and assume that there exists a constant $L > 0$ such that*

$$|W(\xi) - W(\xi')| \leq L|\xi - \xi'| \quad \text{for every } \xi, \xi' \in \mathbb{R}^{3 \times 3}. \quad (3.4)$$

Then for every $\nu \in \mathbb{S}^1$, $\bar{\xi} \in \mathbb{R}^{3 \times 2}$ and $b \in \mathbb{R}^3$,

$$\begin{aligned} \mathcal{Q}^*W(\bar{\xi}|b) &= \inf_{\lambda, \varphi} \left\{ \int_{Q'_\nu \times I} W(\bar{\xi} + \nabla_\alpha \varphi | \lambda \nabla_3 \varphi) dx : \lambda > 0, \varphi \in W^{1,1}(Q'_\nu \times I; \mathbb{R}^3), \right. \\ &\quad \left. \varphi(\cdot, x_3) \text{ is } Q'_\nu\text{-periodic for } \mathcal{L}^1\text{-a.e. } x_3 \in I, \lambda \int_{Q'_\nu \times I} \nabla_3 \varphi dy = b \right\}. \end{aligned}$$

Proof. Fix $\bar{\xi} \in \mathbb{R}^{3 \times 2}$ and $b \in \mathbb{R}^3$, and define for every $\nu \in \mathbb{S}^1$,

$$\begin{aligned} I(\nu) &:= \inf_{\lambda, \varphi} \left\{ \int_{Q'_\nu \times I} W(\bar{\xi} + \nabla_\alpha \varphi | \lambda \nabla_3 \varphi) dx : \lambda > 0, \varphi \in W^{1,1}(Q'_\nu \times I; \mathbb{R}^3), \right. \\ &\quad \left. \varphi(\cdot, x_3) \text{ is } Q'_\nu\text{-periodic for } \mathcal{L}^1\text{-a.e. } x_3 \in I, \lambda \int_{Q'_\nu \times I} \nabla_3 \varphi dy = b \right\}. \end{aligned}$$

We shall prove that for any ν and $\nu' \in \mathbb{S}^1$, then $I(\nu) \leq I(\nu')$. Interchanging the roles of ν and ν' , we will deduce that the inequality is actually an equality, and taking $\nu' = e_2$ that $\mathcal{Q}^*W(\bar{\xi}|b) = I(\nu)$ which is the conclusion of the proposition.

Let $\lambda > 0$ and $\varphi \in W^{1,1}(Q'_\nu \times I; \mathbb{R}^3)$ be such that $\varphi(\cdot, x_3)$ is Q'_ν -periodic for \mathcal{L}^1 -a.e. $x_3 \in I$ and $\lambda \int_{Q'_\nu \times I} \nabla_3 \varphi dy = b$. Extend φ by Q'_ν -periodicity to the whole $\mathbb{R}^2 \times I$ and set $\varphi_n(x_\alpha, x_3) := \varphi(nx_\alpha, x_3)/n$. Consider also a cut-off function $\zeta_k \in C_c^\infty(Q'_\nu; [0, 1])$ satisfying

$$\begin{cases} \zeta_k = 1 \text{ on } Q'_\nu \left(0, 1 - \frac{1}{k}\right), \\ \|\nabla_\alpha \zeta_k\|_{L^\infty(Q'_\nu; \mathbb{R}^2)} \leq 2k^2. \end{cases} \quad (3.5)$$

Define now

$$\psi_{n,k}(x_\alpha, x_3) := \varphi_n(x_\alpha, x_3)\zeta_k(x_\alpha) + \frac{x_3}{\lambda n} \left[b - \lambda n \int_{Q'_\nu \times I} \zeta_k(z_\alpha) \nabla_3 \varphi_n(z_\alpha, z_3) dz \right].$$

It turns out that $\psi_{n,k} \in W^{1,1}(Q'_\nu \times I; \mathbb{R}^3)$, that $\psi_{n,k}(\cdot, x_3)$ is Q'_ν -periodic for \mathcal{L}^1 -a.e. $x_3 \in I$ and that $\lambda n \int_{Q'_\nu \times I} \nabla_3 \psi_{n,k} dy = b$. Hence the pair $(\lambda n, \psi_{n,k})$ is admissible for $I(\nu)$ and thus

$$I(\nu) \leq \int_{Q'_\nu \times I} W(\bar{\xi} + \nabla_\alpha \psi_{n,k} | \lambda n \nabla_3 \psi_{n,k}) dx.$$

Consequently, (3.5) yields to

$$\begin{aligned} I(\nu) &\leq \int_{Q'_\nu \left(0, 1 - \frac{1}{k}\right) \times I} W\left(\bar{\xi} + \nabla_\alpha \varphi_n | \lambda n \nabla_3 \varphi_n + b - \lambda n \int_{Q'_\nu \times I} \zeta_k(z_\alpha) \nabla_3 \varphi_n(z_\alpha, z_3) dz\right) dx \\ &\quad + \int_{(Q'_\nu \setminus Q'_\nu \left(0, 1 - \frac{1}{k}\right)) \times I} W(\bar{\xi} + \nabla_\alpha \psi_{n,k} | \lambda n \nabla_3 \psi_{n,k}) dx \end{aligned}$$

and using the growth condition (H_1) together with the Lipschitz property (3.4) of W , we get that

$$\begin{aligned} I(\nu) &\leq \int_{Q'_\nu \times I} W(\bar{\xi} + \nabla_\alpha \varphi(nx_\alpha, x_3) | \lambda \nabla_3 \varphi(nx_\alpha, x_3)) dx \\ &\quad + \beta \int_{\left(Q'_\nu \setminus Q'_\nu\left(0, 1 - \frac{1}{k}\right)\right) \times I} (1 + |\bar{\xi}| + |\nabla_\alpha \varphi(nx_\alpha, x_3)| + \lambda |\nabla_3 \varphi(nx_\alpha, x_3)| + 2k^2 |\varphi_n(x)|) dx \\ &\quad + L \left| b - \lambda \int_{Q'_\nu \times I} \zeta_k(z_\alpha) \nabla_3 \varphi(nz_\alpha, z_3) dz \right|. \end{aligned}$$

Applying the Riemann-Lebesgue Lemma and the fact that $\varphi_n \rightarrow 0$ in $L^1(Q'_\nu \times I; \mathbb{R}^3)$, it implies, sending $n \rightarrow +\infty$, that

$$\begin{aligned} I(\nu) &\leq \int_{Q'_\nu \times I} W(\bar{\xi} + \nabla_\alpha \varphi(y) | \lambda \nabla_3 \varphi(y)) dy \\ &\quad + \beta \left[1 - \left(1 - \frac{1}{k}\right)^2 \right] \int_{Q'_\nu \times I} (1 + |\bar{\xi}| + |\nabla_\alpha \varphi(y)| + \lambda |\nabla_3 \varphi(y)|) dy \\ &\quad + L \left| b - \left(\lambda \int_{Q'_\nu \times I} \nabla_3 \varphi(z) dz \right) \int_{Q'_\nu} \zeta_k(y_\alpha) dy_\alpha \right|. \end{aligned}$$

As $\lambda \int_{Q'_\nu \times I} \nabla_3 \varphi(z) dz = b$ and $\zeta_k \rightarrow 1$ in $L^1(Q'_\nu)$, we obtain letting $k \rightarrow +\infty$ that

$$I(\nu) \leq \int_{Q'_\nu \times I} W(\bar{\xi} + \nabla_\alpha \varphi(y) | \lambda \nabla_3 \varphi(y)) dy.$$

Taking the infimum over all pairs (λ, φ) as above implies that $I(\nu) \leq I(\nu')$ which is the desired result. \square

3.2 The surface energy density

Let W^∞ (resp. $(\mathcal{Q}^*W)^\infty$) be the recession function of W (resp. \mathcal{Q}^*W) defined by

$$W^\infty(\xi) := \limsup_{t \rightarrow +\infty} \frac{W(t\xi)}{t} \quad \left(\text{resp. } (\mathcal{Q}^*W)^\infty(\xi) := \limsup_{t \rightarrow +\infty} \frac{\mathcal{Q}^*W(t\xi)}{t} \right)$$

for every $\xi \in \mathbb{R}^{3 \times 3}$.

Let $(z, b, \nu) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^1$ and consider $\tau \in \mathbb{S}^1$ such that (τ, ν) is an orthonormal basis of \mathbb{R}^2 . Define the auxiliary surface energy $\gamma : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^1 \rightarrow [0, +\infty)$ by

$$\begin{aligned} \gamma(z, \nu, b) &:= \inf_{\lambda, \varphi} \left\{ \int_{Q'_\nu \times I} W^\infty(\nabla_\alpha \varphi | \lambda \nabla_3 \varphi) dx : \lambda > 0, \varphi \in W^{1,1}(Q'_\nu \times I; \mathbb{R}^3), \varphi^{+\nu} - \varphi^{-\nu} = z, \right. \\ &\quad \left. \varphi \text{ is 1-periodic in the direction } \tau \text{ and } \lambda \int_{Q'_\nu \times I} \nabla_3 \varphi dy = b \right\}, \end{aligned} \quad (3.6)$$

where $\varphi^{\pm\nu}$ stands for the trace of φ on the face $\{(x_\alpha, x_3) \in Q'_\nu : x_\alpha \cdot \nu = \pm 1/2\}$. This density will naturally appear in the proof of the lower bound of the jump part. However, arguing as in [3] page 313, one can observe that γ actually coincides with $(\mathcal{Q}^*W)^\infty$.

Proposition 3.4. *Let $W : \mathbb{R}^{3 \times 3} \rightarrow [0, +\infty)$ be a Borel function satisfying (H_1) , (H_2) and (3.4). Then for every $z, b \in \mathbb{R}^3$ and $\nu \in \mathbb{S}^1$, we have*

$$\gamma(z, \nu, b) = (\mathcal{Q}^*W)^\infty(z \otimes \nu | b) = \mathcal{Q}^*(W^\infty)(z \otimes \nu | b).$$

Proof. The proof is divided into two steps. Firstly we shall prove that $\gamma(z, \nu, b) = \mathcal{Q}^*(W^\infty)(z \otimes \nu | b)$ and then that $\mathcal{Q}^*(W^\infty)(z \otimes \nu | b) = (\mathcal{Q}^*W)^\infty(z \otimes \nu | b)$.

Step 1. Let $\lambda > 0$ and $\psi \in W^{1,1}(Q'_\nu \times I; \mathbb{R}^3)$ be such that $\psi(\cdot, x_3)$ is Q'_ν -periodic for \mathcal{L}^1 -a.e. $x_3 \in I$ and $\lambda \int_{Q'_\nu \times I} \nabla_3 \psi dy = b$. Define

$$\varphi(x_\alpha, x_3) := (x_\alpha \cdot \nu)z + \psi(x_\alpha, x_3) \quad \text{for every } (x_\alpha, x_3) \in Q'_\nu \times I.$$

Clearly $\varphi \in W^{1,1}(Q'_\nu \times I; \mathbb{R}^3)$, φ is 1-periodic in the direction τ and $\varphi^{+\nu} - \varphi^{-\nu} = z$. Moreover, we have that $\lambda \int_{Q'_\nu \times I} \nabla_3 \varphi dy = \lambda \int_{Q'_\nu \times I} \nabla_3 \psi dy = b$. Thus, by (3.6), φ is admissible for $\gamma(z, \nu, b)$ and consequently

$$\gamma(z, \nu, b) \leq \int_{Q'_\nu \times I} W^\infty(\nabla_\alpha \varphi | \lambda \nabla_3 \varphi) dx = \int_{Q'_\nu \times I} W^\infty(z \otimes \nu + \nabla_\alpha \psi | \lambda \nabla_3 \psi) dx.$$

Taking the infimum over all such (λ, ψ) , and using Proposition 3.3 yields $\gamma(z, \nu, b) \leq \mathcal{Q}^*(W^\infty)(z \otimes \nu | b)$.

Conversely, consider $\lambda > 0$ and $\varphi \in W^{1,1}(Q'_\nu \times I; \mathbb{R}^3)$ such that φ is 1-periodic in the direction τ , $\varphi^{+\nu} - \varphi^{-\nu} = z$ and $\lambda \int_{Q'_\nu \times I} \nabla_3 \varphi dy = b$. Define

$$\psi(x_\alpha, x_3) := -(x_\alpha \cdot \nu)z + \varphi(x_\alpha, x_3) \quad \text{for every } (x_\alpha, x_3) \in Q'_\nu \times I.$$

Then $\psi \in W^{1,1}(Q'_\nu \times I; \mathbb{R}^3)$, ψ is 1-periodic in the direction τ . Moreover noticing that $\psi^{+\nu} - \psi^{-\nu} = 0$, it implies that ψ is actually Q'_ν -periodic. As $\lambda \int_{Q'_\nu \times I} \nabla_3 \psi dy = \lambda \int_{Q'_\nu \times I} \nabla_3 \varphi dy = b$ it follows that ψ is admissible for $\mathcal{Q}^*(W^\infty)(z \otimes \nu | b)$ and consequently

$$\mathcal{Q}^*(W^\infty)(z \otimes \nu | b) \leq \int_{Q'_\nu \times I} W^\infty(z \otimes \nu + \nabla_\alpha \psi | \lambda \nabla_3 \psi) dx = \int_{Q'_\nu \times I} W^\infty(\nabla_\alpha \varphi | \lambda \nabla_3 \varphi) dx.$$

By the arbitrariness of (λ, ψ) , it yields $\mathcal{Q}^*(W^\infty)(z \otimes \nu | b) \leq \gamma(z, \nu, b)$ and it completes the proof of the first step.

Step 2. Now take any pair (λ, φ) where $\lambda > 0$ and $\varphi \in W^{1,1}(Q' \times I; \mathbb{R}^3)$ is such that $\varphi(\cdot, x_3)$ is Q' -periodic and $\lambda \int_{Q' \times I} \nabla_3 \varphi dy = b$. Then

$$\frac{\mathcal{Q}^* W(t(z \otimes \nu | b))}{t} \leq \int_{Q' \times I} \frac{W(tz \otimes \nu + \nabla_\alpha(t\varphi) | \lambda \nabla_3(t\varphi))}{t} dx,$$

and by the growth condition (H_1) , we have for $t > 1$,

$$\frac{W(tz \otimes \nu + t \nabla_\alpha \varphi | \lambda t \nabla_3 \varphi)}{t} \leq \beta(1 + |z| + |\nabla_\alpha \varphi| + \lambda |\nabla_3 \varphi|) \in L^1(Q' \times I).$$

Hence by the limsup version of Fatou's Lemma, it follows that

$$\begin{aligned} (\mathcal{Q}^* W)^\infty(z \otimes \nu | b) &= \limsup_{t \rightarrow +\infty} \frac{\mathcal{Q}^* W(t(z \otimes \nu | b))}{t} \leq \limsup_{t \rightarrow +\infty} \int_{Q' \times I} \frac{W(tz \otimes \nu + t \nabla_\alpha \varphi | \lambda t \nabla_3 \varphi)}{t} dx \\ &\leq \int_{Q' \times I} \limsup_{t \rightarrow +\infty} \frac{W(tz \otimes \nu + t \nabla_\alpha \varphi | \lambda t \nabla_3 \varphi)}{t} dx = \int_{Q' \times I} W^\infty(z \otimes \nu + \nabla_\alpha \varphi | \lambda \nabla_3 \varphi) dx. \end{aligned}$$

Finally taking the infimum over all (λ, φ) as before, we obtain that $(\mathcal{Q}^* W)^\infty(z \otimes \nu | b) \leq \mathcal{Q}^*(W^\infty)(z \otimes \nu | b)$.

To prove the converse inequality, for any $t > 1$, let $\lambda_t > 0$ and $\varphi_t \in W^{1,1}(Q' \times I; \mathbb{R}^3)$ be such that $\varphi_t(\cdot, x_3)$ is Q' -periodic for \mathcal{L}^1 -a.e. $x_3 \in I$, $\lambda_t \int_I \nabla_3 \varphi_t dy = b$ and

$$\int_{Q' \times I} W(tz \otimes \nu + t \nabla_\alpha \varphi_t | \lambda_t \nabla_3 \varphi_t) dx \leq \mathcal{Q}^* W(t(z \otimes \nu | b)) + 1. \quad (3.7)$$

By the growth and coercivity properties (H_1) and (3.2), it turns out that

$$\|(\nabla_\alpha \varphi_t | \lambda_t \nabla_3 \varphi_t)\|_{L^1(Q' \times I; \mathbb{R}^{3 \times 3})} \leq C, \quad (3.8)$$

for some constant $C > 0$ independent of t . Hence using (H_2) and the fact that W^∞ is positively 1-homogeneous, it follows that

$$\begin{aligned} \mathcal{Q}^*(W^\infty)(z \otimes \nu | b) &\leq \int_{Q' \times I} W^\infty(z \otimes \nu + \nabla_\alpha \varphi_t | \lambda_t \nabla_3 \varphi_t) dx \\ &\leq \int_{Q' \times I} \frac{W(tz \otimes \nu + t \nabla_\alpha \varphi_t | \lambda_t \nabla_3 \varphi_t)}{t} dx \\ &\quad + \frac{C}{t} \int_{Q' \times I} (1 + t^{1-r} |z|^{1-r} + t^{1-r} |(\nabla_\alpha \varphi_t | \lambda_t \nabla_3 \varphi_t)|^{1-r}) dx. \end{aligned}$$

From Hölder's Inequality together with (3.7) and (3.8), it yields

$$\mathcal{Q}^*(W^\infty)(z \otimes \nu|b) \leq \frac{\mathcal{Q}^*W(t(z \otimes \nu|b))}{t} + \frac{C}{t} + \frac{C}{t^r}.$$

Finally, taking the limsup as $t \rightarrow +\infty$ leads to $\mathcal{Q}^*(W^\infty)(z \otimes \nu|b) \leq (\mathcal{Q}^*W)^\infty(z \otimes \nu|b)$ which concludes the proof of the second step and of the proposition. \square

4 Properties of the Γ -limit

We start by localizing the functionals on \mathcal{A}_0 , the family of all bounded open subsets of \mathbb{R}^2 . Let $J_\varepsilon : BV(\mathbb{R}^3; \mathbb{R}^3) \times \mathcal{M}(\mathbb{R}^2; \mathbb{R}^3) \times \mathcal{A}_0 \rightarrow [0, +\infty]$ be defined by

$$J_\varepsilon(u, \bar{b}, A) := \begin{cases} \int_{A \times I} W\left(\nabla_\alpha u \Big| \frac{1}{\varepsilon} \nabla_3 u\right) dx & \text{if } \begin{cases} u \in W^{1,1}(A \times I; \mathbb{R}^3), \\ \bar{b} = \frac{1}{\varepsilon} \int_I \nabla_3 u(\cdot, x_3) dx_3, \end{cases} \\ +\infty & \text{otherwise.} \end{cases} \quad (4.1)$$

In the sequel, we will also use the family $\mathcal{A}(\omega)$ of all open subsets of ω . For every sequence $\{\varepsilon_j\} \searrow 0^+$ define the Γ -lower limit of J_{ε_j} given by

$$J_{\{\varepsilon_j\}}(u, \bar{b}, A) := \inf_{\{u_j, \bar{b}_j\}} \left\{ \liminf_{j \rightarrow +\infty} J_{\varepsilon_j}(u_j, \bar{b}_j, A) : u_j \overset{*}{\rightharpoonup} u \text{ in } BV(A \times I; \mathbb{R}^3), \bar{b}_j \overset{*}{\rightharpoonup} \bar{b} \text{ in } \mathcal{M}(A; \mathbb{R}^3) \right\}.$$

In order to show that the family $\{J_\varepsilon\}$ Γ -converges to the functional E , it is enough to prove that for every sequence $\{\varepsilon_j\} \searrow 0^+$, there exists a further subsequence $\{\varepsilon_{j_n}\}$ such that $J_{\{\varepsilon_{j_n}\}}(u, \bar{b}, \omega) = E(u, \bar{b})$ for any $(u, \bar{b}) \in BV(\omega; \mathbb{R}^3) \times \mathcal{M}(\omega; \mathbb{R}^3)$.

It is easily seen from the coercivity condition (H_1) that if $J_{\{\varepsilon_j\}}(u, \bar{b}, \omega) < +\infty$, then necessarily $D_3 u = 0$ so that u (may be identified to a function which) belongs to $BV(\omega; \mathbb{R}^3)$. Thus it suffices to consider $(u, \bar{b}) \in BV(\omega; \mathbb{R}^3) \times \mathcal{M}(\omega; \mathbb{R}^3)$ in which case we have that

$$\begin{aligned} J_{\{\varepsilon_j\}}(u, \bar{b}, A) &= \inf_{\{u_j\}} \left\{ \liminf_{j \rightarrow +\infty} \int_{A \times I} W\left(\nabla_\alpha u_j \Big| \frac{1}{\varepsilon_j} \nabla_3 u_j\right) dx : \{u_j\} \subset W^{1,1}(A \times I; \mathbb{R}^3) \right. \\ &\quad \left. u_j \overset{*}{\rightharpoonup} u \text{ in } BV(A \times I; \mathbb{R}^3), \frac{1}{\varepsilon_j} \int_I \nabla_3 u_j(\cdot, x_3) dx_3 \overset{*}{\rightharpoonup} \bar{b} \text{ in } \mathcal{M}(A; \mathbb{R}^3) \right\}. \end{aligned} \quad (4.2)$$

Note that thanks to the coercivity condition (H_1) , the weak* convergence in $BV(A \times I; \mathbb{R}^3)$ in (4.2) is equivalent to the strong convergence in $L^1(A \times I; \mathbb{R}^3)$.

It is expected, as in most variational problems in BV (see [15]), that the Γ -limit should be the sum of three terms relative to the decomposition of the gradient $D_\alpha u$ of a function $u \in BV(\omega; \mathbb{R}^3)$ into bulk, jump and Cantor parts. In the present study, there will be a fourth one which comes from the presence of the bending moment \bar{b} , and which is due to the fact that \bar{b} may be singular with respect to $D_\alpha u$. There is no hope to avoid this so called singular term as the following example shows.

Example 4.1. There exist $(u, \bar{b}) \in BV(\omega; \mathbb{R}^3) \times \mathcal{M}(\omega; \mathbb{R}^3)$ and a sequence $\{u_\varepsilon\} \subset W^{1,1}(\Omega; \mathbb{R}^3)$ such that $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $BV(\Omega; \mathbb{R}^3)$, $\frac{1}{\varepsilon} \int_I \nabla_3 u_\varepsilon(\cdot, x_3) dx_3 \overset{*}{\rightharpoonup} \bar{b}$ in $\mathcal{M}(\omega; \mathbb{R}^3)$ where the measures $D_\alpha u$ and \bar{b} are mutually singular.

For simplicity, we construct an example for scalar valued functions. Consider a nonnegative radial function $\varrho \in C_c^\infty(\mathbb{R}^3)$ such that $\text{Supp}(\varrho) \subset B(0, 1/2)$ and $\int_{\mathbb{R}^3} \varrho(x) dx = 1$, and set $\varphi(x_\alpha, x_3) := \int_{-1/2}^{x_3} \varrho(x_\alpha, s) ds$. Assume that ω contains the origin and define $u_\varepsilon \in W^{1,1}(\Omega)$ by

$$u_\varepsilon(x_\alpha, x_3) := u(x_\alpha) + \frac{1}{\varepsilon} \varphi\left(\frac{x}{\varepsilon}\right),$$

where $u \in W^{1,1}(\omega)$. Then, by a change of variables, we have

$$\|u_\varepsilon - u\|_{L^1(\Omega)} \leq \varepsilon, \quad \|\nabla u_\varepsilon\|_{L^1(\Omega; \mathbb{R}^3)} \leq \|\nabla u\|_{L^1(\omega; \mathbb{R}^3)} + \varepsilon \|\nabla \varphi\|_{L^1(\Omega; \mathbb{R}^3)}$$

so that $u_\varepsilon \rightharpoonup u$ in $W^{1,1}(\Omega)$ (and thus also weakly* in $BV(\Omega)$). On the other hand, we have that

$$\frac{1}{\varepsilon} \nabla_3 u_\varepsilon(x) = \frac{1}{\varepsilon^3} \varrho\left(\frac{x}{\varepsilon}\right)$$

and consequently, $\frac{1}{\varepsilon} \int_I \nabla_3 u_\varepsilon(\cdot, x_3) dx_3 \xrightarrow{*} \delta$ in $\mathcal{M}(\omega)$, where δ is the Dirac mass at $0 \in \mathbb{R}^2$, which is singular with respect to $D_\alpha u = \nabla_\alpha u \mathcal{L}^2$.

Remark 4.2. In [9, Theorem 1.2], it has been shown that

$$J_{\{\varepsilon_j\}}(u, \bar{b}, \omega) = \int_\omega \mathcal{Q}^* W(\nabla_\alpha u | \bar{b}) dx_\alpha = E(u, \bar{b}),$$

for $u \in W^{1,p}(\omega; \mathbb{R}^3)$ and $\bar{b} \in L^p(\omega; \mathbb{R}^3)$ with $p > 1$. An analogous argument ensures that the same representation result holds when $u \in W^{1,1}(\omega; \mathbb{R}^3)$ and $\bar{b} \in L^1(\omega; \mathbb{R}^3)$.

Remark 4.3. Arguing exactly as in [9, Lemma 2.3], one can show that $J_{\{\varepsilon_j\}}$ remains unchanged if we replace W by its quasiconvexification $\mathcal{Q}W$ in (4.1). Hence using (3.3), upon replacing W by $\mathcal{Q}W$, we may assume without loss of generality that W is quasiconvex. Then, by the growth condition (H_1) and *e.g.* the proof of Theorem 2.1, Step 2 in [20], there exists a constant $L > 0$ such that

$$|W(\xi) - W(\xi')| \leq L|\xi - \xi'|, \quad (4.3)$$

for every ξ and $\xi' \in \mathbb{R}^{3 \times 3}$. As a consequence, W^∞ is Lipschitz continuous as well and

$$|W^\infty(\xi) - W^\infty(\xi')| \leq L|\xi - \xi'|. \quad (4.4)$$

Let \mathcal{R}_0 be the countable subfamily of \mathcal{A}_0 obtained by taking all finite unions of open squares in \mathbb{R}^2 with faces parallel to the axes, centered at $x_\alpha \in \mathbb{Q}^2$, and with rational edge length. Since $\mathcal{M}(\omega; \mathbb{R}^3)$ and $BV(\Omega; \mathbb{R}^3)$ are the duals of separable spaces (see *e.g.* [3, Remark 3.12]), the adopted weak* topologies in (4.2), and their metrizability on bounded sets, ensure the applicability of Kuratowsky's Compactness Theorem (we refer to *e.g.* [13, Corollary 8.12] for the weak topology of a Banach space with separable dual; it can be checked that a similar result holds for the weak* topology of a Banach space which is the dual of a separable one). Thus, through a diagonal argument, it guarantees the existence of a subsequence $\{\varepsilon_n\} \equiv \{\varepsilon_{j_n}\}$ of $\{\varepsilon_j\}$ such that $J_{\{\varepsilon_n\}}(u, \bar{b}, A)$ is the Γ -limit of $J_{\varepsilon_n}(u, \bar{b}, A)$ for all $A \in \mathcal{R}_0$ (and also $A = \omega$) and all (u, \bar{b}) in $BV(A; \mathbb{R}^3) \times \mathcal{M}(A; \mathbb{R}^3)$.

Lemma 4.4. *Let $\omega \subset \mathbb{R}^2$ be a bounded open set and let $A \subset \subset \omega$ be an open subset of ω with Lipschitz boundary. For every $(u, \bar{b}) \in BV(\omega; \mathbb{R}^3) \times \mathcal{M}(\omega; \mathbb{R}^3)$ satisfying $|\bar{b}|(\partial A) = 0$, there exists a sequence $\{v_n\} \subset W^{1,1}(A \times I; \mathbb{R}^3)$ such that*

$$\begin{cases} v_n \rightarrow u \text{ in } L^1(A \times I; \mathbb{R}^3), \\ \frac{1}{\varepsilon_n} \int_I \nabla_3 v_n(\cdot, x_3) dx_3 \xrightarrow{*} \bar{b} \text{ in } \mathcal{M}(A; \mathbb{R}^3), \\ T v_n = T u \text{ on } \partial A \times I, \\ |D_\alpha v_n|(A \times I) \rightarrow |D_\alpha u|(A), \\ \frac{1}{\varepsilon_n} |D_3 v_n|(A \times I) \rightarrow |\bar{b}|(A). \end{cases}$$

Proof. By [8, Lemma 2.5], there exists a sequence $\{\tilde{v}_n\} \subset W^{1,1}(A; \mathbb{R}^3)$ such that $\tilde{v}_n \rightarrow u$ in $L^1(A; \mathbb{R}^3)$, $|D_\alpha \tilde{v}_n|(A) \rightarrow |D_\alpha u|(A)$ and $T \tilde{v}_n = T u$ on ∂A . Consider a usual sequence of mollifiers denoted by $\{\varrho_k\}$. Then from [3, Theorem 2.2], we have that $\bar{b} * \varrho_k \xrightarrow{*} \bar{b}$ in $\mathcal{M}_{\text{loc}}(\omega; \mathbb{R}^3)$ and thus

$$\bar{b} * \varrho_k \xrightarrow{*} \bar{b} \text{ in } \mathcal{M}(A; \mathbb{R}^3). \quad (4.5)$$

Moreover, since $|\bar{b}|(\partial A) = 0$, it follows that $|\bar{b} * \varrho_k|(A) \rightarrow |\bar{b}|(A)$. As $\bar{b} * \varrho_k \in L^1(A; \mathbb{R}^3)$ one can find $\bar{b}_k \in \mathcal{C}_c^\infty(A; \mathbb{R}^3)$ such that

$$\|\bar{b}_k - (\bar{b} * \varrho_k)\|_{L^1(A; \mathbb{R}^3)} \leq \frac{1}{k}. \quad (4.6)$$

Now define

$$v_n^k(x_\alpha, x_3) := \tilde{v}_n(x_\alpha) + \varepsilon_n x_3 \bar{b}_k(x_\alpha).$$

The sequence $\{v_n^k\} \subset W^{1,1}(A \times I; \mathbb{R}^3)$, $v_n^k \rightarrow u$ in $L^1(A \times I; \mathbb{R}^3)$ as $n \rightarrow +\infty$ and $Tv_n^k = Tu$ on $\partial A \times I$. Moreover from the lower semicontinuity of the total variation, we infer that

$$\lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} |D_\alpha v_n^k|(A \times I) = |D_\alpha u|(A)$$

and from (4.5) and (4.6),

$$\frac{1}{\varepsilon_n} \int_I \nabla_3 v_n^k(\cdot, x_3) dx_3 = \bar{b}_k \xrightarrow[k \rightarrow +\infty]{*} \bar{b} \text{ in } \mathcal{M}(A; \mathbb{R}^3),$$

uniformly with respect to $n \in \mathbb{N}$. Using the separability of $\mathcal{C}_0(A; \mathbb{R}^3)$ and a diagonalization argument (see *e.g.* [12, Lemma 7.1]), one may find a sequence $k(n) \nearrow +\infty$ such that, setting $v_n := v_n^{k(n)}$, then $v_n \rightarrow u$ in $L^1(A \times I; \mathbb{R}^3)$, $\frac{1}{\varepsilon_n} \int_I \nabla_3 v_n(\cdot, x_3) dx_3 \xrightarrow{*} \bar{b}$ in $\mathcal{M}(A; \mathbb{R}^3)$, $Tv_n = Tu$ on $\partial A \times I$ for all $n \in \mathbb{N}$, $|D_\alpha v_n|(A \times I) \rightarrow |D_\alpha u|(A)$ and $\frac{1}{\varepsilon_n} |D_3 v_n|(A \times I) = |\bar{b}_{k(n)}|(A) \rightarrow |\bar{b}|(A)$. \square

Using Lemma 4.4 and an adaptation of the proof of [9, Lemma 2.2], we can prove the following result which will be instrumental in the proof of the lower bound. It states that, without loss of generality, recovery sequences can be taken in such a way to match the lateral boundary of their target.

Lemma 4.5. *Let $\omega \subset \mathbb{R}^2$ be a bounded open set and let $A \subset\subset \omega$ be an open subset with Lipschitz boundary. Consider $(u, \bar{b}) \in BV(\omega; \mathbb{R}^3) \times \mathcal{M}(\omega; \mathbb{R}^3)$ such that $|\bar{b}|(\partial A) = 0$ and assume that $\{u_n\} \subset W^{1,1}(A \times I; \mathbb{R}^3)$ is a sequence satisfying $u_n \rightarrow u$ in $L^1(A \times I; \mathbb{R}^3)$, $\frac{1}{\varepsilon_n} \int_I \nabla_3 u_n(\cdot, x_3) dx_3 \xrightarrow{*} \bar{b}$ in $\mathcal{M}(A; \mathbb{R}^3)$ and*

$$\lim_{n \rightarrow +\infty} \int_{A \times I} W\left(\nabla_\alpha u_n \left| \frac{1}{\varepsilon_n} \nabla_3 u_n\right.\right) dx = \ell,$$

for some $\ell > 0$. Then there exist a subsequence $\{n_k\} \nearrow +\infty$ and a sequence $\{v_k\} \subset W^{1,1}(A \times I; \mathbb{R}^3)$ satisfying $Tv_k = Tu$ on $\partial A \times I$, $v_k \rightarrow u$ in $L^1(A \times I; \mathbb{R}^3)$, $\frac{1}{\varepsilon_{n_k}} \int_I \nabla_3 v_k(\cdot, x_3) dx_3 \xrightarrow{} \bar{b}$ in $\mathcal{M}(A; \mathbb{R}^3)$, and*

$$\limsup_{k \rightarrow +\infty} \int_{A \times I} W\left(\nabla_\alpha v_k \left| \frac{1}{\varepsilon_{n_k}} \nabla_3 v_k\right.\right) dx \leq \ell.$$

Remark 4.6. If $u \in W^{1,1}(\omega; \mathbb{R}^3)$ then by [9, Lemma 2.2] it is not necessary to assume neither ∂A to be Lipschitz nor that $|\bar{b}|(\partial A) = 0$. In that case the conclusion is that $v_k = u$ on a neighborhood of $\partial A \times I$.

To prove the upper bound, we will also need the following locality result.

Lemma 4.7. *Let $\omega \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary and let $W : \mathbb{R}^{3 \times 3} \rightarrow [0, +\infty)$ be a Borel function satisfying (H_1) . For every $(u, \bar{b}) \in BV(\omega; \mathbb{R}^3) \times \mathcal{M}(\omega; \mathbb{R}^3)$, the set function $J_{\{\varepsilon_n\}}(u, \bar{b}, \cdot)$ is the trace on $\mathcal{A}(\omega)$ of a Radon measure absolutely continuous with respect to $\mathcal{L}^2 + |D_\alpha u| + |\bar{b}|$.*

Proof. Fix $(u, \bar{b}) \in BV(\omega; \mathbb{R}^3) \times \mathcal{M}(\omega; \mathbb{R}^3)$. Since ω has a Lipschitz boundary, the extension of u by zero outside ω is a $BV(\mathbb{R}^2; \mathbb{R}^3)$. Hence upon extending u and \bar{b} by zero outside ω , we may assume without loss of generality that $\bar{b} \in \mathcal{M}(\mathbb{R}^2; \mathbb{R}^3)$ and $u \in BV(\mathbb{R}^2; \mathbb{R}^3)$.

Assume first that $A \in \mathcal{A}_0$, that ∂A is Lipschitz and satisfies $|\bar{b}|(\partial A) = 0$. By Lemma 4.4, taking $\{v_n\}$ as test function for $J_{\{\varepsilon_n\}}(u, \bar{b}, A)$ and using the growth condition (H_1) , we get that

$$0 \leq J_{\{\varepsilon_n\}}(u, \bar{b}, A) \leq \beta(\mathcal{L}^2(A) + |D_\alpha u|(A) + |\bar{b}|(A)).$$

Consider now an arbitrary open set $A \in \mathcal{A}(\omega)$. By [13, Example 14.8], for any $\eta > 0$, there exists an open set C with smooth boundary such that $A \subset\subset C$ and

$$\mathcal{L}^2(C \setminus A) + |D_\alpha u|(C \setminus A) + |\bar{b}|(C \setminus A) < \eta/\beta. \quad (4.7)$$

Note that C may not be contained in ω and this is the reason why we need to extend u and \bar{b} outside ω . By [18, Lemma 14.16], the function $x \mapsto \text{dist}(x, \partial C)$ is smooth on a suitable δ -neighborhood of ∂C for some $\delta < \text{dist}(A, \partial C)$. For every $t \in [0, \delta]$, define

$$C_t := \{x \in C : \text{dist}(x, \partial C) > t\} \quad \text{and} \quad S_t := \{x \in C : \text{dist}(x, \partial C) = t\}.$$

As the family $\{S_t\}_t$ is made of pairwise disjoint sets, it is possible to find $t_0 \in [0, \delta]$ such that $\bar{b}|(S_{t_0}) = 0$. Since $S_{t_0} = \partial C_{t_0}$, it follows that C_{t_0} is a smooth open set satisfying $A \subset \subset C_{t_0} \subset C$. Since $J_{\{\varepsilon_n\}}(u, \bar{b}, \cdot)$ is an increasing set function, we obtain from the first case together with (4.7) that

$$\begin{aligned} J_{\{\varepsilon_n\}}(u, \bar{b}, A) &\leq J_{\{\varepsilon_n\}}(u, \bar{b}, C_{t_0}) \leq \beta(\mathcal{L}^2(C_{t_0}) + |D_\alpha u|(C_{t_0}) + |\bar{b}|(C_{t_0})) \\ &\leq \beta(\mathcal{L}^2(A) + |D_\alpha u|(A) + |\bar{b}|(A)) + \eta \end{aligned}$$

and the thesis comes from the arbitrariness of η . Repeating word for word the proof of [9, Lemma 2.1], we get that $J_{\{\varepsilon_n\}}(u, \bar{b}, \cdot)$ is the restriction to $\mathcal{A}(\omega)$ of a Radon measure absolutely continuous with respect to $\mathcal{L}^2 + |D_\alpha u| + |\bar{b}|$. Note that there is no need to extract a further subsequence as stated in [9] since we already did it passing from $\{\varepsilon_j\}$ to $\{\varepsilon_n\} \equiv \{\varepsilon_{j_n}\}$. \square

5 Proof of the lower bound

Lemma 5.1. *For every $(u, \bar{b}) \in BV(\omega; \mathbb{R}^3) \times \mathcal{M}(\omega; \mathbb{R}^3)$, then $J_{\{\varepsilon_n\}}(u, \bar{b}, \omega) \geq E(u, \bar{b})$.*

Proof. Fix $(u, \bar{b}, A) \in BV(\omega; \mathbb{R}^3) \times \mathcal{M}(\omega; \mathbb{R}^3) \times \mathcal{A}(\omega)$. Thanks to the Besicovitch Decomposition Theorem, one may find four mutually singular measures $\bar{b}^a, \bar{b}^j, \bar{b}^c$ and \bar{b}^σ such that $\bar{b} = \bar{b}^a + \bar{b}^j + \bar{b}^c + \bar{b}^\sigma$ and $\bar{b}^a \ll \mathcal{L}^2$, $\bar{b}^j \ll \mathcal{H}^1 \llcorner J_u$ and $\bar{b}^c \ll |D_\alpha^c u|$.

Consider a sequence $\{u_n\} \subset W^{1,1}(\Omega; \mathbb{R}^3)$ such that $u_n \xrightarrow{*} u$ in $BV(\Omega; \mathbb{R}^3)$, $\frac{1}{\varepsilon_n} \int_I \nabla_3 u_n(\cdot, x_3) dx_3 \xrightarrow{*} \bar{b}$ in $\mathcal{M}(\omega; \mathbb{R}^3)$, and

$$J_{\{\varepsilon_n\}}(u, \bar{b}, \omega) = \lim_{n \rightarrow +\infty} \int_\Omega W \left(\nabla_\alpha u_n \middle| \frac{1}{\varepsilon_n} \nabla_3 u_n \right) dx.$$

For every Borel set $B \subset \omega$, define

$$\mu_n(B) := \int_{B \times I} W \left(\nabla_\alpha u_n \middle| \frac{1}{\varepsilon_n} \nabla_3 u_n \right) dx \quad \text{and} \quad \bar{b}_n(B) := \frac{1}{\varepsilon_n} \int_{B \times I} \nabla_3 u_n dx.$$

It turns out that $\{\mu_n\}$ and $\{|\bar{b}_n|\}$ are sequences of nonnegative Radon measures uniformly bounded in $\mathcal{M}(\omega)$. Hence we can extract subsequences, still denoted $\{\mu_n\}$ and $\{|\bar{b}_n|\}$, and find μ and $\lambda \in \mathcal{M}(\omega)$ so that $\mu_n \xrightarrow{*} \mu$ and $|\bar{b}_n| \xrightarrow{*} \lambda$ in $\mathcal{M}(\omega)$. Similarly we can decompose the measure μ as the sum of five mutually singular measures $\mu^a, \mu^j, \mu^c, \mu^\sigma$ and μ^s such that $\mu = \mu^a + \mu^j + \mu^c + \mu^\sigma + \mu^s$ and $\mu^a \ll \mathcal{L}^2$, $\mu^j \ll \mathcal{H}^1 \llcorner J_u$, $\mu^c \ll |D_\alpha^c u|$ and $\mu^\sigma \ll |\bar{b}^\sigma|$.

Since $\mu(\omega) \leq J_{\{\varepsilon_n\}}(u, \bar{b}, \omega)$, in order to show the lower bound, it is enough to check that $\mu(\omega) \geq E(u, \bar{b})$ or that

$$\frac{d\mu^a}{d\mathcal{L}^2}(x_0) \geq \mathcal{Q}^* W \left(\nabla_\alpha u(x_0) \middle| \frac{d\bar{b}}{d\mathcal{L}^2}(x_0) \right) \quad \text{for } \mathcal{L}^2\text{-a.e. } x_0 \in \omega, \quad (5.1)$$

$$\frac{d\mu^j}{d\mathcal{H}^1 \llcorner J_u}(x_0) \geq (\mathcal{Q}^* W)^\infty \left((u^+(x_0) - u^-(x_0)) \otimes \nu_u(x_0), \frac{d\bar{b}}{d\mathcal{H}^1 \llcorner J_u}(x_0) \right) \quad \text{for } \mathcal{H}^1\text{-a.e. } x_0 \in J_u, \quad (5.2)$$

$$\frac{d\mu^c}{d|D_\alpha^c u|}(x_0) \geq (\mathcal{Q}^* W)^\infty \left(\frac{dD_\alpha u}{d|D_\alpha^c u|}(x_0) \middle| \frac{d\bar{b}}{d|D_\alpha^c u|}(x_0) \right) \quad \text{for } |D_\alpha^c u|\text{-a.e. } x_0 \in \omega, \quad (5.3)$$

$$\frac{d\mu^\sigma}{d|\bar{b}^\sigma|}(x_0) \geq (\mathcal{Q}^* W)^\infty \left(0 \middle| \frac{d\bar{b}}{d|\bar{b}^\sigma|}(x_0) \right) \quad \text{for } |\bar{b}^\sigma|\text{-a.e. } x_0 \in \omega. \quad (5.4)$$

Indeed, if the four previous properties hold, we obtain that

$$\begin{aligned} &\int_\omega \mathcal{Q}^* W \left(\nabla_\alpha u \middle| \frac{d\bar{b}}{d\mathcal{L}^2} \right) dx + \int_{J_u} (\mathcal{Q}^* W)^\infty \left((u^+ - u^-) \otimes \nu_u, \frac{d\bar{b}}{d\mathcal{H}^1 \llcorner J_u} \right) d\mathcal{H}^1 \\ &+ \int_\omega (\mathcal{Q}^* W)^\infty \left(\frac{dD_\alpha u}{d|D_\alpha^c u|} \middle| \frac{d\bar{b}}{d|D_\alpha^c u|} \right) d|D_\alpha^c u| + \int_\omega (\mathcal{Q}^* W)^\infty \left(0 \middle| \frac{d\bar{b}}{d|\bar{b}^\sigma|} \right) d|\bar{b}^\sigma| \\ &= \mu^a(\omega) + \mu^j(\omega) + \mu^c(\omega) + \mu^\sigma(\omega) \leq \mu(\omega) \leq J_{\{\varepsilon_n\}}(u, \bar{b}, \omega), \end{aligned}$$

which is the announced claim. \square

The remaining of the section is devoted to prove the inequalities (5.1)-(5.4)

Proof of (5.1). Let $x_0 \in \omega$ be such that the Radon-Nikodým derivative of μ and \bar{b} at x_0 with respect to \mathcal{L}^2 exist and are finite, which is also a Lebesgue point for u , $\nabla_\alpha u$ and $\frac{d\bar{b}}{d\mathcal{L}^2}$, a point of approximate differentiability of u , and

$$\frac{d|\mu - \mu^a|}{d\mathcal{L}^2}(x_0) = \frac{d|\bar{b} - \bar{b}^a|}{d\mathcal{L}^2}(x_0) = 0. \quad (5.5)$$

Note that since $|\bar{b} - \bar{b}^a|$ and $|\mu - \mu^a|$ are singular with respect to the Lebesgue measure, then \mathcal{L}^2 almost every points $x_0 \in \omega$ satisfy these properties. Let $\{\rho_k\}$ be a sequence converging to zero and such that $\lambda(\partial Q'(x_0, \rho_k)) = \mu(\partial Q'(x_0, \rho_k)) = 0$ for every $k \in \mathbb{N}$. Hence it follows from (5.5) that

$$\begin{aligned} \frac{d\mu^a}{d\mathcal{L}^2}(x_0) &= \frac{d\mu}{d\mathcal{L}^2}(x_0) = \lim_{k \rightarrow +\infty} \frac{\mu(Q'(x_0, \rho_k))}{\rho_k^2} \\ &= \lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} \frac{1}{\rho_k^2} \int_{Q'(x_0, \rho_k) \times I} W \left(\nabla_\alpha u_n \Big| \frac{1}{\varepsilon_n} \nabla_3 u_n \right) dx \\ &= \lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} \int_{Q' \times I} W \left(\nabla_\alpha u_n(x_0 + \rho_k y_\alpha, y_3) \Big| \frac{1}{\varepsilon_n} \nabla_3 u_n(x_0 + \rho_k y_\alpha, y_3) \right) dy \\ &= \lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} \int_{Q' \times I} W \left(\nabla_\alpha u_{n,k} \Big| \frac{\rho_k}{\varepsilon_n} \nabla_3 u_{n,k} \right) dy, \end{aligned} \quad (5.6)$$

where we set $u_{n,k}(y_\alpha, y_3) := [u_n(x_0 + \rho_k y_\alpha, y_3) - u(x_0)]/\rho_k$.

Since x_0 is a point of approximate differentiability of u and $u_n \rightarrow u$ in $L^1(\Omega; \mathbb{R}^3)$, defining $u_0(y_\alpha, y_3) := \nabla_\alpha u(x_0)y_\alpha$, it results that

$$\lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} \|u_{n,k} - u_0\|_{L^1(Q' \times I; \mathbb{R}^3)} = 0. \quad (5.7)$$

On the other hand, using (5.5), the fact that $(1/\varepsilon_n) \int_I \nabla_3 u_n(\cdot, x_3) dx_3 \xrightarrow{*} \bar{b}$ in $\mathcal{M}(\omega; \mathbb{R}^3)$ and that x_0 is a Lebesgue point of $\frac{d\bar{b}}{d\mathcal{L}^2}$, for every $\varphi \in \mathcal{C}_0(Q'; \mathbb{R}^3)$ we get that

$$\lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} \int_{Q'} \left(\frac{\rho_k}{\varepsilon_n} \int_I \nabla_3 u_{n,k}(y_\alpha, y_3) dy_3 \right) \cdot \varphi(y_\alpha) dy_\alpha = \frac{d\bar{b}}{d\mathcal{L}^2}(x_0) \cdot \int_{Q'} \varphi(y_\alpha) dy_\alpha. \quad (5.8)$$

Moreover, since $\lambda(\partial Q'(x_0, \rho_k)) = 0$ for each $k \in \mathbb{N}$, [3, Proposition 1.62] ensures that $\bar{b}_n(Q'(x_0, \rho_k)) \rightarrow \bar{b}(Q'(x_0, \rho_k))$ as $n \rightarrow +\infty$ and thus

$$\lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} \frac{\rho_k}{\varepsilon_n} \int_{Q' \times I} \nabla_3 u_{n,k} dy = \frac{d\bar{b}}{d\mathcal{L}^2}(x_0). \quad (5.9)$$

Gathering (5.6), (5.7), (5.8) and (5.9) and using the fact that $\mathcal{M}(Q'; \mathbb{R}^3)$ is the dual of the separable space $\mathcal{C}_0(Q'; \mathbb{R}^3)$, by means of a standard diagonalization process, one may construct a sequence $\bar{u}_k := u_{n_k, k} - u_0$ and $\delta_k := \varepsilon_{n_k}/\rho_k$ such that $\bar{u}_k \rightarrow 0$ in $L^1(Q' \times I; \mathbb{R}^3)$, $\delta_k \rightarrow 0$, $(1/\delta_k) \int_I \nabla_3 \bar{u}_k(\cdot, y_3) dy_3 \xrightarrow{*} \frac{d\bar{b}}{d\mathcal{L}^2}(x_0) \mathcal{L}^2$ in $\mathcal{M}(Q'; \mathbb{R}^3)$,

$$\frac{1}{\delta_k} \int_{Q' \times I} \nabla_3 \bar{u}_k dy \rightarrow \frac{d\bar{b}}{d\mathcal{L}^2}(x_0). \quad (5.10)$$

and

$$\frac{d\mu^a}{d\mathcal{L}^2}(x_0) = \lim_{k \rightarrow +\infty} \int_{Q' \times I} W \left(\nabla_\alpha u(x_0) + \nabla_\alpha \bar{u}_k \Big| \frac{1}{\delta_k} \nabla_3 \bar{u}_k \right) dy. \quad (5.11)$$

Using Remark 4.6, one may assume without loss of generality that $\bar{u}_k = 0$ on a neighborhood of $\partial Q' \times I$. We now define

$$\varphi_k(x_\alpha, x_3) := \bar{u}_k(x_\alpha, x_3) + \delta_k x_3 \left(\frac{d\bar{b}}{d\mathcal{L}^2}(x_0) - \frac{1}{\delta_k} \int_{Q' \times I} \nabla_3 \bar{u}_k(y) dy \right).$$

Then, $\varphi_k \in W^{1,1}(Q' \times I; \mathbb{R}^3)$, $\varphi_k(\cdot, x_3)$ is Q' -periodic for \mathcal{L}^1 -a.e. $x_3 \in I$ and

$$\frac{1}{\delta_k} \int_{Q' \times I} \nabla_3 \varphi_k dy = \frac{d\bar{b}}{d\mathcal{L}^2}(x_0).$$

Hence φ_k is an admissible test function for $\mathcal{Q}^*W\left(\nabla_\alpha u(x_0)\left|\frac{d\bar{b}}{d\mathcal{L}^2}(x_0)\right.\right)$, and using (5.11) together with the Lipschitz property (4.3), we get that

$$\begin{aligned}\frac{d\mu^a}{d\mathcal{L}^2}(x_0) &\geq \limsup_{k \rightarrow +\infty} \int_{Q' \times I} W\left(\nabla_\alpha u(x_0) + \nabla_\alpha \varphi_k \left| \frac{1}{\delta_k} \nabla_3 \varphi_k \right.\right) dy \\ &\quad - L \limsup_{k \rightarrow +\infty} \left| \frac{d\bar{b}}{d\mathcal{L}^2}(x_0) - \frac{1}{\delta_k} \int_{Q' \times I} \nabla_3 \bar{u}_k dy \right|.\end{aligned}$$

Relation (5.10) enables us to conclude that the last term in the previous inequality is actually zero and thus

$$\frac{d\mu^a}{d\mathcal{L}^2}(x_0) \geq \mathcal{Q}^*W\left(\nabla_\alpha u(x_0)\left|\frac{d\bar{b}}{d\mathcal{L}^2}(x_0)\right.\right).$$

Proof of (5.2). Let $x_0 \in J_u$, then there exists $u^-(x_0), u^+(x_0) \in \mathbb{R}^3$ (with $u^-(x_0) \neq u^+(x_0)$) and $\nu = \nu_u(x_0) \in \mathbb{S}^1$ such that

$$\lim_{\rho \rightarrow 0^+} \frac{1}{\rho^2} \int_{\{y_\alpha \in Q'_\nu(x_0, \rho) : \pm(y_\alpha - x_0) \cdot \nu > 0\}} |u(y_\alpha) - u^\pm(x_0)| dy_\alpha = 0.$$

Assume that the Radon-Nikodým derivative of μ and \bar{b} at x_0 with respect to $\mathcal{H}^1 \llcorner J_u$ exist and are finite, that x_0 is Lebesgue point for $\frac{d\bar{b}}{d\mathcal{H}^1 \llcorner J_u}$ with respect to $\mathcal{H}^1 \llcorner J_u$, that

$$\frac{d|\mu - \mu^j|}{d\mathcal{H}^1 \llcorner J_u}(x_0) = \frac{d|\bar{b} - \bar{b}^j|}{d\mathcal{H}^1 \llcorner J_u}(x_0) = 0, \quad (5.12)$$

and

$$\lim_{\rho \rightarrow 0^+} \frac{\mathcal{H}^1(J_u \cap Q'_\nu(x_0, \rho))}{\rho} = 1. \quad (5.13)$$

Assume further that $\pi_\nu := \nu^\perp$ is the approximate tangent space of J_u at x_0 , i.e.,

$$\lim_{\rho \rightarrow 0} \frac{1}{\rho} \int_{J_u} \phi\left(\frac{x_\alpha - x_0}{\rho}\right) d\mathcal{H}^1(x_\alpha) = \int_{\pi_\nu} \phi(x_\alpha) d\mathcal{H}^1(x_\alpha) \quad \text{for all } \phi \in \mathcal{C}_c(\mathbb{R}^2). \quad (5.14)$$

Note that \mathcal{H}^1 almost every points x_0 in J_u satisfy the preceding requirements. Indeed (5.13) is a consequence of the countably \mathcal{H}^1 -rectifiability of J_u (see e.g. [3, Theorem 2.63]), property (5.12) is due to the fact that the measures $|\mu - \mu^j|$ and $|\bar{b} - \bar{b}^j|$ are singular with respect to $\mathcal{H}^1 \llcorner J_u$ while (5.14) is a consequence of the Federer-Vol'pert Theorem (see [3, Theorem 3.78]).

Let $\{\rho_k\} \searrow 0^+$ be such that $\lambda(\partial Q'_\nu(x_0, \rho_k)) = \mu(\partial Q'_\nu(x_0, \rho_k)) = 0$ for each $k \in \mathbb{N}$. Then by virtue of (5.12) and (5.13), we infer that

$$\begin{aligned}\frac{d\mu^j}{d\mathcal{H}^1 \llcorner J_u}(x_0) &= \frac{d\mu}{d\mathcal{H}^1 \llcorner J_u}(x_0) = \lim_{k \rightarrow +\infty} \frac{\mu(Q'_\nu(x_0, \rho_k))}{\mathcal{H}^1(Q'_\nu(x_0, \rho_k) \cap J_u)} = \lim_{k \rightarrow +\infty} \frac{\mu(Q'_\nu(x_0, \rho_k))}{\rho_k} \\ &= \lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} \frac{1}{\rho_k} \int_{Q'_\nu(x_0, \rho_k) \times I} W\left(\nabla_\alpha u_n \left| \frac{1}{\varepsilon_n} \nabla_3 u_n \right.\right) dx \\ &= \lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} \rho_k \int_{Q'_\nu \times I} W\left(\nabla_\alpha u_n(x_0 + \rho_k y_\alpha, y_3) \left| \frac{1}{\varepsilon_n} \nabla_3 u_n(x_0 + \rho_k y_\alpha, y_3) \right.\right) dy \\ &= \lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} \rho_k \int_{Q'_\nu \times I} W\left(\frac{1}{\rho_k} \left(\nabla_\alpha v_{n,k} \left| \frac{\rho_k}{\varepsilon_n} \nabla_3 v_{n,k} \right.\right)\right) dy,\end{aligned} \quad (5.15)$$

where $v_{n,k}(y) := u_n(x_0 + \rho_k y_\alpha, y_3)$. Set

$$v_0(y) := \begin{cases} u^+(x_0) & \text{if } y_\alpha \cdot \nu > 0 \\ u^-(x_0) & \text{if } y_\alpha \cdot \nu \leq 0. \end{cases}$$

As $x_0 \in J_u$ and $u_n \rightarrow u$ in $L^1(\Omega; \mathbb{R}^3)$, it follows that

$$\lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} \|v_{n,k} - v_0\|_{L^1(Q'_\nu \times I; \mathbb{R}^3)} = 0. \quad (5.16)$$

Since $(1/\varepsilon_n) \int_I \nabla_3 u_n(\cdot, x_3) dx_3 \xrightarrow{*} \bar{b}$ in $\mathcal{M}(\omega; \mathbb{R}^3)$, we deduce that for every $\varphi \in \mathcal{C}_0(Q'_\nu; \mathbb{R}^3)$,

$$\lim_{n \rightarrow +\infty} \int_{Q'_\nu} \left(\frac{\rho_k}{\varepsilon_n} \int_I \nabla_3 v_{n,k}(y_\alpha, y_3) dy_3 \right) \cdot \varphi(y_\alpha) dy_\alpha = \frac{1}{\rho_k} \int_{Q'_\nu(x_0, \rho_k)} \varphi \left(\frac{x_\alpha - x_0}{\rho_k} \right) \cdot d\bar{b}(x_\alpha).$$

Using the fact that x_0 is a Lebesgue point of $\frac{d\bar{b}}{d\mathcal{H}^1 \llcorner J_u}$ together with (5.12), (5.13) and (5.14) we infer that

$$\lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} \int_{Q'_\nu} \left(\frac{\rho_k}{\varepsilon_n} \int_I \nabla_3 v_{n,k}(y_\alpha, y_3) dy_3 \right) \cdot \varphi(y_\alpha) dy_\alpha = \frac{d\bar{b}}{d\mathcal{H}^1 \llcorner J_u}(x_0) \cdot \int_{\pi_\nu} \varphi d\mathcal{H}^1. \quad (5.17)$$

Moreover, as for the bulk part, using the fact that $\lambda(\partial Q'_\nu(x_0, \rho_k)) = 0$ for every $k \in \mathbb{N}$, we have that

$$\lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} \frac{\rho_k}{\varepsilon_n} \int_{Q'_\nu \times I} \nabla_3 v_{n,k} dy = \frac{d\bar{b}}{d\mathcal{H}^1 \llcorner J_u}(x_0). \quad (5.18)$$

Using again the separability of $\mathcal{C}_0(Q'_\nu; \mathbb{R}^3)$ together with a diagonalization argument, from (5.15), (5.16), (5.17) and (5.18), we obtain the existence of sequences $\bar{v}_k := v_{n_k, k} \in W^{1,1}(Q'_\nu \times I; \mathbb{R}^3)$ and $\delta_k := \varepsilon_{n_k}/\rho_k$ with the properties that $\delta_k \rightarrow 0$, $\bar{v}_k \rightarrow v_0$ in $L^1(Q'_\nu \times I; \mathbb{R}^3)$, $(1/\delta_k) \int_I \nabla_3 \bar{v}_k(\cdot, x_3) dx_3 \xrightarrow{*} \frac{d\bar{b}}{d\mathcal{H}^1 \llcorner J_u}(x_0) \mathcal{H}^1 \llcorner \pi_\nu$ in $\mathcal{M}(Q'_\nu; \mathbb{R}^3)$,

$$\frac{1}{\delta_k} \int_{Q'_\nu \times I} \nabla_3 \bar{v}_k dy \rightarrow \frac{d\bar{b}}{d\mathcal{H}^1 \llcorner J_u}(x_0), \quad (5.19)$$

and

$$\frac{d\mu^j}{d\mathcal{H}^1 \llcorner J_u}(x_0) = \lim_{k \rightarrow +\infty} \rho_k \int_{Q'_\nu \times I} W \left(\frac{1}{\rho_k} \left(\nabla_\alpha \bar{v}_k \middle| \frac{1}{\delta_k} \nabla_3 \bar{v}_k \right) \right) dy.$$

By the coercivity condition (H_1) and the previous relation, it follows that the sequence of scaled gradients $\{(\nabla_\alpha \bar{v}_k | (1/\delta_k) \nabla_3 \bar{v}_k)\}$ is uniformly bounded in $L^1(Q'_\nu \times I; \mathbb{R}^{3 \times 3})$. Thus, using (H_2) and the fact that the recession function W^∞ is positively 1-homogeneous, we obtain that

$$\begin{aligned} & \rho_k \int_{Q'_\nu \times I} \left| W \left(\frac{1}{\rho_k} \left(\nabla_\alpha \bar{v}_k \middle| \frac{1}{\delta_k} \nabla_3 \bar{v}_k \right) \right) - W^\infty \left(\frac{1}{\rho_k} \left(\nabla_\alpha \bar{v}_k \middle| \frac{1}{\delta_k} \nabla_3 \bar{v}_k \right) \right) \right| dy \\ & \leq C \int_{Q'_\nu \times I} \left(\rho_k + \rho_k^r \left| \left(\nabla_\alpha \bar{v}_k \middle| \frac{1}{\delta_k} \nabla_3 \bar{v}_k \right) \right|^{1-r} \right) dy \\ & \leq C \rho_k + C \rho_k^r \|(\nabla_\alpha \bar{v}_k | (1/\delta_k) \nabla_3 \bar{v}_k)\|_{L^1(Q'_\nu \times I; \mathbb{R}^{3 \times 3})}^{1-r} \rightarrow 0 \end{aligned}$$

where we applied Hölder's Inequality. As a consequence

$$\frac{d\mu^j}{d\mathcal{H}^1 \llcorner J_u}(x_0) = \lim_{k \rightarrow +\infty} \int_{Q'_\nu \times I} W^\infty \left(\nabla_\alpha \bar{v}_k \middle| \frac{1}{\delta_k} \nabla_3 \bar{v}_k \right) dy.$$

Since $\mathcal{H}^1(\pi_\nu \cap \partial Q'_\nu) = 0$, we can apply Lemma 4.5 (with W^∞ instead of W) so that, up to an extraction, there is no loss of generality to assume that $T\bar{v}_k = Tv_0$. Define

$$\varphi_k(x_\alpha, x_3) := \bar{v}_k(x_\alpha, x_3) + \delta_k x_3 \left(\frac{d\bar{b}}{d\mathcal{H}^1 \llcorner J_u}(x_0) - \frac{1}{\delta_k} \int_{Q'_\nu \times I} \nabla_3 \bar{v}_k(y) dy \right),$$

and denote by $\varphi_k^{\pm\nu}$ the trace of φ_k on the faces $\{(x_\alpha, x_3) \in Q'_\nu \times I : x_\alpha \cdot \nu = \pm 1/2\}$. Then $\varphi_k \in W^{1,1}(Q'_\nu \times I; \mathbb{R}^3)$ is 1-periodic in the direction τ (where $\tau \in \mathbb{S}^1$ is such that (τ, ν) is an orthonormal basis of \mathbb{R}^2), $\varphi_k^{+\nu} - \varphi_k^{-\nu} = u^+(x_0) - u^-(x_0)$ and

$$\frac{1}{\delta_k} \int_{Q'_\nu \times I} \nabla_3 \varphi_k dy = \frac{d\bar{b}}{d\mathcal{H}^1 \llcorner J_u}(x_0).$$

In particular, φ_k is an admissible test function for $\gamma\left(u^+(x_0) - u^-(x_0), \nu_u(x_0), \frac{d\bar{b}}{d\mathcal{H}^1 \llcorner J_u}(x_0)\right)$ and using the Lipschitz condition (4.4) satisfied by W^∞ , we infer that

$$\begin{aligned} \frac{d\mu^j}{d\mathcal{H}^1 \llcorner J_u}(x_0) &\geq \limsup_{k \rightarrow +\infty} \int_{Q'_\nu \times I} W^\infty \left(\nabla_\alpha \varphi_k \left| \frac{1}{\delta_k} \nabla_3 \varphi_k \right. \right) dy \\ &\quad - L \limsup_{k \rightarrow +\infty} \left| \frac{d\bar{b}}{d\mathcal{H}^1 \llcorner J_u}(x_0) - \frac{1}{\delta_k} \int_{Q'_\nu \times I} \nabla_3 \bar{v}_k dy \right|. \end{aligned}$$

From (5.19), it follows that the last term of the previous relation is actually zero. Hence

$$\frac{d\mu^j}{d\mathcal{H}^1 \llcorner J_u}(x_0) \geq \gamma \left(u^+(x_0) - u^-(x_0), \nu_u(x_0), \frac{d\bar{b}}{d\mathcal{H}^1 \llcorner J_u}(x_0) \right),$$

and consequently by virtue of (4.3) and Proposition 3.4 it results that

$$\frac{d\mu^j}{d\mathcal{H}^1 \llcorner J_u}(x_0) \geq (\mathcal{Q}^* W)^\infty \left((u^+(x_0) - u^-(x_0)) \otimes \nu_u(x_0), \frac{d\bar{b}}{d\mathcal{H}^1 \llcorner J_u}(x_0) \right).$$

Proof of (5.3). Fix a point $x_0 \in \omega$ such that the matrix

$$A(x_0) := \frac{dD_\alpha u}{d|D_\alpha u|}(x_0) \text{ has rank one and } |A(x_0)| = 1, \quad (5.20)$$

the Radon-Nikodým derivative of μ and \bar{b} with respect to $|D_\alpha^c u|$ exist and are finite,

$$\frac{d|\mu - \mu^c|}{d|D_\alpha^c u|}(x_0) = \frac{d|\bar{b} - \bar{b}^c|}{d|D_\alpha^c u|}(x_0) = 0, \quad (5.21)$$

$$\frac{d|D_\alpha u|}{d|D_\alpha^c u|}(x_0) = 1 \quad (5.22)$$

and

$$\lim_{\rho \rightarrow 0^+} \frac{|D_\alpha u|(Q'(x_0, \rho))}{\rho} = 0, \quad \lim_{\rho \rightarrow 0^+} \frac{|D_\alpha u|(Q'(x_0, \rho))}{\rho^2} = +\infty. \quad (5.23)$$

Assume also that for every $t \in (0, 1)$,

$$\liminf_{\rho \rightarrow 0^+} \frac{|D_\alpha u|(Q'(x_0, \rho) \setminus Q'(x_0, t\rho))}{|D_\alpha u|(Q'(x_0, \rho))} \leq 1 - t^2. \quad (5.24)$$

Note that $|D_\alpha^c u|$ almost every points x_0 in ω satisfy these properties. Indeed, (5.20) is a consequence of Alberti's Rank One Theorem (see [1]); properties (5.21) come from the fact that $|\mu - \mu^c|$ and $|\bar{b} - \bar{b}^c|$ are singular with respect to $|D_\alpha^c u|$; property (5.22) is due to the Besicovitch Differentiation Theorem; both relations in (5.23) are obtained from [3, Proposition 3.92] and finally, property (5.24) is proved in [15, Lemma 2.13].

Since $A(x_0)$ has rank one, there exists $a \in \mathbb{R}^3$ and $\nu \in \mathbb{S}^1$ such that $A(x_0) := a \otimes \nu$. We may assume without loss of generality that $\nu = e_2$.

Fix $t \in (0, 1)$ arbitrarily close to 1 and thanks to (5.24), choose a sequence $\{\rho_k\} \searrow 0^+$ such that

$$\limsup_{k \rightarrow +\infty} \frac{|D_\alpha u|(Q'(x_0, \rho_k) \setminus Q'(x_0, t\rho_k))}{|D_\alpha u|(Q'(x_0, \rho_k))} \leq 1 - t^2. \quad (5.25)$$

Moreover, up to a subsequence (still denoted $\{\rho_k\}$), it is possible to find $\nu^c \in \text{Tan}(|D_\alpha^c u|, x_0)$ (depending on t), *i.e.*,

$$\lim_{k \rightarrow +\infty} \frac{1}{|D_\alpha^c u|(Q'(x_0, \rho_k))} \int_{\mathbb{R}^2} \varphi \left(\frac{x_\alpha - x_0}{\rho_k} \right) d|D_\alpha^c u|(x_\alpha) = \int_{\mathbb{R}^2} \varphi(x_\alpha) d\nu^c(x_\alpha), \quad (5.26)$$

for all $\varphi \in \mathcal{C}_c(\mathbb{R}^2)$. Fix $\gamma \in (t, 1)$, then by (5.21) and (5.22),

$$\begin{aligned} \frac{d\mu^c}{d|D_\alpha^c u|}(x_0) &= \frac{d\mu}{d|D_\alpha^c u|}(x_0) = \lim_{k \rightarrow +\infty} \frac{\mu(Q'(x_0, \rho_k))}{|D_\alpha^c u|(Q'(x_0, \rho_k))} = \lim_{k \rightarrow +\infty} \frac{\mu(Q'(x_0, \rho_k))}{|D_\alpha u|(Q'(x_0, \rho_k))} \\ &\geq \limsup_{k \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \frac{1}{|D_\alpha u|(Q'(x_0, \rho_k))} \int_{Q'(x_0, \gamma \rho_k) \times I} W \left(\nabla_\alpha u_n \Big| \frac{1}{\varepsilon_n} \nabla_3 u_n \right) dx. \end{aligned} \quad (5.27)$$

Define

$$\begin{cases} z_k(x_\alpha) := \frac{\rho_k}{|D_\alpha u|(Q'(x_0, \rho_k))} \left[u(x_0 + \rho_k x_\alpha) - \int_{Q'} u(x_0 + \rho_k y_\alpha) dy_\alpha \right], \\ w_{n,k}(x_\alpha, x_3) := \frac{\rho_k}{|D_\alpha u|(Q'(x_0, \rho_k))} \left[u_n(x_0 + \rho_k x_\alpha, x_3) - \int_{Q' \times I} u_n(x_0 + \rho_k y_\alpha, y_3) dy \right]. \end{cases}$$

Changing variable in (5.27) and setting

$$t_k := \frac{|D_\alpha u|(Q'(x_0, \rho_k))}{\rho_k^2},$$

we get that

$$\frac{d\mu^c}{d|D_\alpha^c u|}(x_0) \geq \limsup_{k \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \frac{1}{t_k} \int_{(\gamma Q') \times I} W \left(t_k \left(\nabla_\alpha w_{n,k} \Big| \frac{\rho_k}{\varepsilon_n} \nabla_3 w_{n,k} \right) \right) dx. \quad (5.28)$$

Using the fact that $u_n \rightarrow u$ in $L^1(\Omega; \mathbb{R}^3)$ we obtain

$$\lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} \|w_{n,k} - z_k\|_{L^1(Q' \times I; \mathbb{R}^3)} = 0. \quad (5.29)$$

As $\int_{Q'} z_k dx_\alpha = 0$ and $|D_\alpha z_k|(Q') = 1$, it follows that the sequence $\{z_k\}$ is relatively compact in $L^1(Q'; \mathbb{R}^3)$ and by [3, Theorem 3.95], any limit function w is representable by

$$w(x_\alpha) = a \theta(x_2)$$

for some increasing function $\theta \in BV(-1/2, 1/2)$ (recall that we assumed $\nu = e_2$). Hence, using (5.29) it follows that

$$\lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} \|w_{n,k} - w\|_{L^1(Q' \times I; \mathbb{R}^3)} = 0. \quad (5.30)$$

Now take $\varphi \in \mathcal{C}_0(Q'; \mathbb{R}^3)$, then changing variables using the fact that $(1/\varepsilon_n) \int_I \nabla_3 u_n(\cdot, y_3) dy_3 \xrightarrow{*} \bar{b}$ in $\mathcal{M}(\omega; \mathbb{R}^3)$ together with (5.21), (5.22) and (5.26), it follows that

$$\lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} \int_{Q'} \varphi(x_\alpha) \cdot \left(\frac{\rho_k}{\varepsilon_n} \int_I \nabla_3 w_{n,k}(x_\alpha, x_3) dx_3 \right) dx_\alpha = \frac{\bar{d}\bar{b}}{d|D_\alpha^c u|}(x_0) \cdot \int_{Q'} \varphi(x_\alpha) d\nu^c(x_\alpha). \quad (5.31)$$

Gathering (5.28), (5.30) and (5.31), the separability of $\mathcal{C}_0(Q'; \mathbb{R}^3)$ together with a standard diagonalization argument, it leads to the existence of a subsequence $n_k \nearrow +\infty$ such that, setting $\bar{w}_k := w_{n_k, k}$ and $\delta_k := \varepsilon_{n_k}/\rho_k$, then $\delta_k \searrow 0^+$, $\bar{w}_k \rightarrow w$ in $L^1(Q' \times I; \mathbb{R}^3)$, $\frac{1}{\delta_k} \int_I \nabla_3 \bar{w}_k(\cdot, x_3) dx_3 \xrightarrow{*} \frac{\bar{d}\bar{b}}{d|D_\alpha^c u|}(x_0) \nu^c$ in $\mathcal{M}(Q'; \mathbb{R}^3)$ and

$$\frac{d\mu^c}{d|D_\alpha^c u|}(x_0) \geq \limsup_{k \rightarrow +\infty} \frac{1}{t_k} \int_{(\gamma Q') \times I} W \left(t_k \left(\nabla_\alpha \bar{w}_k \Big| \frac{1}{\delta_k} \nabla_3 \bar{w}_k \right) \right) dx. \quad (5.32)$$

We may also assume without loss of generality that

$$\left| \frac{1}{\delta_k} \int_I \nabla_3 \bar{w}_k(\cdot, x_3) dx_3 \right| \xrightarrow{*} \lambda^c \quad \text{in } \mathcal{M}(Q')$$

for some non negative Radon measure $\lambda^c \in \mathcal{M}(Q')$.

Thanks to the coercivity condition (H_1) , the sequence of scaled gradients $\{(\nabla_\alpha \bar{w}_k | (1/\delta_k) \nabla_3 \bar{w}_k)\}$ is uniformly bounded in $L^1((\gamma Q') \times I; \mathbb{R}^{3 \times 3})$. Thus using hypothesis (H_2) and Hölder's Inequality, we get that

$$\begin{aligned} \frac{1}{t_k} \int_{(\gamma Q') \times I} \left| W^\infty \left(t_k \left(\nabla_\alpha \bar{w}_k \left| \frac{1}{\delta_k} \nabla_3 \bar{w}_k \right. \right) \right) - W \left(t_k \left(\nabla_\alpha \bar{w}_k \left| \frac{1}{\delta_k} \nabla_3 \bar{w}_k \right. \right) \right) \right| dx \\ \leq \frac{C}{t_k} + \frac{C}{t_k^r} \int_{(\gamma Q') \times I} \left| \left(\nabla_\alpha \bar{w}_k \left| \frac{1}{\delta_k} \nabla_3 \bar{w}_k \right. \right) \right|^{1-r} dx \\ \leq \frac{C}{t_k} + \frac{C}{t_k^r} \|(\nabla_\alpha \bar{w}_k | (1/\delta_k) \nabla_3 \bar{w}_k)\|_{L^1((\gamma Q') \times I; \mathbb{R}^{3 \times 3})}^{1-r} \rightarrow 0, \end{aligned}$$

where we used the fact that, thanks to (5.23), $t_k \rightarrow +\infty$. But as W^∞ is positively 1-homogeneous, we get from (5.32)

$$\frac{d\mu^c}{d|D_\alpha^c u|}(x_0) \geq \limsup_{k \rightarrow +\infty} \int_{(\gamma Q') \times I} W^\infty \left(\nabla_\alpha \bar{w}_k \left| \frac{1}{\delta_k} \nabla_3 \bar{w}_k \right. \right) dx.$$

Extend θ continuously to \mathbb{R} by the value of its trace at $\pm 1/2$. Let ϱ_k be a usual sequence of (one dimensional) mollifiers and set

$$\tilde{w}_k(x_\alpha, x_3) := a(\theta * \varrho_k)(x_2) + \delta_k x_3 \frac{d\bar{b}}{d|D_\alpha^c u|}(x_0).$$

Then $\tilde{w}_k \in W^{1,1}(Q' \times I; \mathbb{R}^3)$, $\tilde{w}_k \rightarrow w$ in $L^1(Q' \times I; \mathbb{R}^3)$ and $\frac{1}{\delta_k} \int_I \nabla_3 \tilde{w}_k dx_3 = \frac{d\bar{b}}{d|D_\alpha^c u|}(x_0)$ for each $k \in \mathbb{N}$. Thus $z_k - \tilde{w}_k \rightarrow 0$ in $L^1(Q' \times I; \mathbb{R}^3)$ and

$$D_\alpha z_k((sQ') \times I) - D_\alpha \tilde{w}_k((sQ') \times I) \rightarrow 0 \quad (5.33)$$

except at most for countably many $s \in (0, 1)$. Fix $s \in (t, \gamma)$ so that (5.33) holds and $\lambda^c(\partial(sQ')) = 0$. Using a standard cut-off function argument, we may assume without loss of generality that $\bar{w}_k = \tilde{w}_k$ on a neighborhood of $\partial(sQ') \times I$ and

$$\frac{d\mu^c}{d|D_\alpha^c u|}(x_0) \geq \limsup_{k \rightarrow +\infty} \int_{(sQ') \times I} W^\infty \left(\nabla_\alpha \bar{w}_k \left| \frac{1}{\delta_k} \nabla_3 \bar{w}_k \right. \right) dx. \quad (5.34)$$

We now compute

$$D_\alpha z_k(sQ') = \frac{D_\alpha u(Q'(x_0, s\rho_k))}{|D_\alpha u|(Q'(x_0, \rho_k))} \quad \text{and} \quad D_\alpha \tilde{w}_k((sQ') \times I) = sA_k \quad (5.35)$$

where

$$A_k := a \otimes e_2[(\theta * \varrho_k)(s/2) - (\theta * \varrho_k)(-s/2)].$$

Note that by (5.20), (5.25), (5.33) and (5.35), we have that

$$\begin{aligned} \limsup_{k \rightarrow +\infty} |sA_k - A(x_0)| &= \limsup_{k \rightarrow +\infty} |D_\alpha \tilde{w}_k((sQ') \times I) - A(x_0)| \\ &= \limsup_{k \rightarrow +\infty} |D_\alpha z_k((sQ') \times I) - A(x_0)| \\ &= \limsup_{k \rightarrow +\infty} \left| \frac{D_\alpha u(Q'(x_0, s\rho_k))}{|D_\alpha u|(Q'(x_0, \rho_k))} - A(x_0) \right| \\ &\leq \limsup_{k \rightarrow +\infty} \frac{|D_\alpha u|(Q'(x_0, \rho_k) \setminus Q'(x_0, s\rho_k))}{|D_\alpha u|(Q'(x_0, \rho_k))} \\ &\quad + \limsup_{k \rightarrow +\infty} \left| \frac{D_\alpha u(Q'(x_0, \rho_k))}{|D_\alpha u|(Q'(x_0, \rho_k))} - A(x_0) \right| \leq 1 - t^2. \end{aligned} \quad (5.36)$$

We now define our last sequence

$$\varphi_k(x_\alpha, x_3) := \bar{w}_k(sx_\alpha, x_3) - sA_k x_\alpha + \delta_k x_3 \left(\frac{d\bar{b}}{d|D_\alpha^c u|}(x_0) - \frac{1}{\delta_k} \int_{Q' \times I} \nabla_3 \bar{w}_k(sy_\alpha, y_3) dy \right).$$

As $\bar{w}_k = \tilde{w}_k$ on $\partial(sQ') \times I$ and \tilde{w}_k depends only on (x_2, x_3) , it is clear from the definition of A_k that φ_k is 1-periodic in the direction e_1 . A simple computation shows that for a.e. x_1 and $x_3 \in I$, then $\varphi_k(x_1, -1/2, x_3) = \varphi_k(x_1, 1/2, x_3)$ so that φ_k is also 1-periodic in the e_2 direction. Moreover we have that

$$\frac{1}{\delta_k} \int_{Q' \times I} \nabla_3 \varphi_k(y) dy = \frac{d\bar{b}}{d|D_\alpha^c u|}(x_0).$$

Hence using (5.34), the Lipschitz condition (4.4) satisfied by W^∞ and a change of variable, we obtain that

$$\begin{aligned} \frac{d\mu^c}{d|D_\alpha^c u|}(x_0) &\geq \limsup_{k \rightarrow +\infty} s \int_{Q' \times I} W^\infty \left(A(x_0) + \nabla_\alpha \varphi_k \left| \frac{s}{\delta_k} \nabla_3 \varphi_k \right| \right) dx \\ &\quad - L s^2 \limsup_{k \rightarrow +\infty} \left| \frac{d\bar{b}}{d|D_\alpha^c u|}(x_0) - \frac{1}{\delta_k s^2} \int_{(sQ') \times I} \nabla_3 \bar{w}_k(y) dy \right| - L s \limsup_{k \rightarrow +\infty} |s A_k - A(x_0)|. \end{aligned}$$

But since $\lambda^c(\partial(sQ')) = 0$, it follows that

$$\frac{1}{\delta_k} \int_{(sQ') \times I} \nabla_3 \bar{w}_k dx \rightarrow \frac{d\bar{b}}{d|D_\alpha^c u|}(x_0) \nu^c(sQ')$$

and from (5.36) that

$$\frac{d\mu^c}{d|D_\alpha^c u|}(x_0) \geq s \mathcal{Q}^*(W^\infty) \left(A(x_0) \left| \frac{d\bar{b}}{d|D_\alpha^c u|}(x_0) \right| \right) - L \left| \frac{d\bar{b}}{d|D_\alpha^c u|}(x_0) \right| |s^2 - \nu^c(sQ')| - L s (1 - t^2). \quad (5.37)$$

From (5.25) we infer that $\nu^c(sQ') \geq \nu^c(\overline{tQ'}) \geq t^2$ and thus

$$t^2 - s^2 \leq \nu^c(sQ') - s^2 \leq 1 - s^2.$$

Passing to the limit first as $s \rightarrow 1^-$ and then as $t \rightarrow 1^-$, we deduce from (5.37) that

$$\frac{d\mu^c}{d|D_\alpha^c u|}(x_0) \geq \mathcal{Q}^*(W^\infty) \left(A(x_0) \left| \frac{d\bar{b}}{d|D_\alpha^c u|}(x_0) \right| \right)$$

and (5.3) follows from Proposition 3.4.

Proof of (5.4). Let $x_0 \in \omega$ be such that the Radon-Nikodým derivative of μ and \bar{b} at x_0 with respect to $|\bar{b}^\sigma|$ exist and are finite, such that

$$\frac{d|\mu - \mu^\sigma|}{d|\bar{b}^\sigma|}(x_0) = \frac{d|\bar{b} - \bar{b}^\sigma|}{d|\bar{b}^\sigma|}(x_0) = 0, \quad (5.38)$$

and such that

$$\frac{d\mathcal{L}^2}{d|\bar{b}^\sigma|}(x_0) = \frac{d|D_\alpha u|}{d|\bar{b}^\sigma|}(x_0) = 0. \quad (5.39)$$

Note that since $|\bar{b}^\sigma|$ is singular with respect to \mathcal{L}^2 , $|D_\alpha u|$, $|\mu - \mu^\sigma|$ and $|\bar{b} - \bar{b}^\sigma|$, it turns out that $|\bar{b}^\sigma|$ almost every points x_0 in ω satisfy these properties.

Let $\{\rho_k\} \searrow 0^+$ be such that $\lambda(\partial Q'(x_0, \rho_k)) = \mu(\partial Q'(x_0, \rho_k)) = 0$ for each $k \in \mathbb{N}$ and extract eventually a subsequence (still denoted $\{\rho_k\}$) such that there exists $\nu^\sigma \in \text{Tan}(|\bar{b}^\sigma|, x_0)$, i.e.,

$$\lim_{k \rightarrow +\infty} \frac{1}{|\bar{b}^\sigma|(Q'(x_0, \rho_k))} \int_{\mathbb{R}^2} \varphi \left(\frac{x_\alpha - x_0}{\rho_k} \right) d\bar{b}^\sigma(x_\alpha) = \int_{\mathbb{R}^2} \varphi(x_\alpha) d\nu^\sigma(x_\alpha) \quad \text{for all } \varphi \in \mathcal{C}_c(\mathbb{R}^2). \quad (5.40)$$

Then by (5.38) and a change of variables

$$\begin{aligned} \frac{d\mu^\sigma}{d|\bar{b}^\sigma|}(x_0) &= \frac{d\mu}{d|\bar{b}^\sigma|}(x_0) = \lim_{k \rightarrow +\infty} \frac{\mu(Q'(x_0, \rho_k))}{|\bar{b}^\sigma|(Q'(x_0, \rho_k))} \\ &= \lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} \frac{1}{t_k} \int_{Q' \times I} W \left(\nabla_\alpha u_n(x_0 + \rho_k y_\alpha, y_3) \left| \frac{1}{\varepsilon_n} \nabla_3 u_n(x_0 + \rho_k y_\alpha, y_3) \right| \right) dy, \end{aligned} \quad (5.41)$$

where

$$t_k := \frac{|\bar{b}^\sigma|(Q'(x_0, \rho_k))}{\rho_k^2}.$$

Define

$$\begin{cases} \psi_{n,k}(x_\alpha, x_3) := \frac{\rho_k}{|\bar{b}^\sigma|(Q'(x_0, \rho_k))} \left[u_n(x_0 + \rho_k x_\alpha, x_3) - \int_{Q' \times I} u_n(x_0 + \rho_k y_\alpha, y_3) dy \right], \\ \psi_k(x_\alpha) := \frac{\rho_k}{|\bar{b}^\sigma|(Q'(x_0, \rho_k))} \left[u(x_0 + \rho_k x_\alpha) - \int_{Q'} u(x_0 + \rho_k y_\alpha) dy_\alpha \right]. \end{cases}$$

Replacing in (5.41), we get that

$$\frac{d\mu^\sigma}{d|\bar{b}^\sigma|}(x_0) = \lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} \frac{1}{t_k} \int_{Q' \times I} W \left(t_k \left(\nabla_\alpha \psi_{n,k} \middle| \frac{\rho_k}{\varepsilon_n} \nabla_3 \psi_{n,k} \right) \right) dx. \quad (5.42)$$

Using the fact that $u_n \rightarrow u$ in $L^1(\Omega; \mathbb{R}^3)$ we obtain that $\psi_{n,k} \rightarrow \psi_k$ in $L^1(Q' \times I; \mathbb{R}^3)$ as $n \rightarrow +\infty$. Moreover, as $\int_{Q'} \psi_k dx_\alpha = 0$ and by (5.39),

$$|D_\alpha \psi_k|(Q') = \frac{|D_\alpha u|(Q'(x_0, \rho_k))}{|\bar{b}^\sigma|(Q'(x_0, \rho_k))} \rightarrow 0,$$

the Poincaré-Wirtinger Inequality implies that $\psi_k \rightarrow 0$ in $L^1(Q'; \mathbb{R}^3)$, hence

$$\lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} \|\psi_{n,k}\|_{L^1(Q' \times I; \mathbb{R}^3)} = 0. \quad (5.43)$$

Consider $\varphi \in \mathcal{C}_0(Q'; \mathbb{R}^3)$, then changing variables and using the fact that $(1/\varepsilon_n) \int_I \nabla_3 u_n(\cdot, y_3) dy_3 \xrightarrow{*} \bar{b}$ in $\mathcal{M}(\omega; \mathbb{R}^3)$ together with (5.38) and (5.40), it follows that

$$\lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} \int_{Q'} \varphi(x_\alpha) \cdot \left(\frac{\rho_k}{\varepsilon_n} \int_I \nabla_3 \psi_{n,k}(x_\alpha, x_3) dx_3 \right) dx_\alpha = \frac{d\bar{b}}{d|\bar{b}^\sigma|}(x_0) \cdot \int_{Q'} \varphi d\nu^\sigma. \quad (5.44)$$

Moreover since $\lambda(\partial Q'(x_0, \rho_k)) = 0$ for every $k \in \mathbb{N}$, we deduce that

$$\lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} \frac{\rho_k}{\varepsilon_n} \int_{Q' \times I} \nabla_3 \psi_{n,k} dy = \frac{d\bar{b}}{d|\bar{b}^\sigma|}(x_0). \quad (5.45)$$

Gathering (5.42), (5.43), (5.44) and (5.45), and using the separability of $\mathcal{C}_0(Q'; \mathbb{R}^3)$ together with a standard diagonalization argument leads to the existence of a subsequence $\{n_k\} \nearrow +\infty$ such that, setting $\phi_k := \psi_{n_k, k}$ and $\delta_k := \varepsilon_{n_k}/\rho_k$, then $\delta_k \searrow 0^+$, $\phi_k \rightarrow 0$ in $L^1(Q' \times I; \mathbb{R}^3)$, $\frac{1}{\delta_k} \int_I \nabla_3 \phi_k(\cdot, x_3) dx_3 \xrightarrow{*} \frac{d\bar{b}}{d|\bar{b}^\sigma|}(x_0) \nu^\sigma$ in $\mathcal{M}(Q'; \mathbb{R}^3)$,

$$\frac{1}{\delta_k} \int_{Q' \times I} \nabla_3 \phi_k dy \rightarrow \frac{d\bar{b}}{d|\bar{b}^\sigma|}(x_0) \quad (5.46)$$

and

$$\frac{d\mu^\sigma}{d|\bar{b}^\sigma|}(x_0) = \lim_{k \rightarrow +\infty} \frac{1}{t_k} \int_{Q' \times I} W \left(t_k \left(\nabla_\alpha \phi_k \middle| \frac{1}{\delta_k} \nabla_3 \phi_k \right) \right) dx. \quad (5.47)$$

By virtue of the coercivity condition (H_1) , the sequence of scaled gradients $\{(\nabla_\alpha \phi_k | (1/\delta_k) \nabla_3 \phi_k)\}$ is uniformly bounded in $L^1(Q' \times I; \mathbb{R}^{3 \times 3})$. Thus using hypothesis (H_2) and Hölder's Inequality, we get that

$$\begin{aligned} & \frac{1}{t_k} \int_{Q' \times I} \left| W^\infty \left(t_k \left(\nabla_\alpha \phi_k \middle| \frac{1}{\delta_k} \nabla_3 \phi_k \right) \right) - W \left(t_k \left(\nabla_\alpha \phi_k \middle| \frac{1}{\delta_k} \nabla_3 \phi_k \right) \right) \right| dx \\ & \leq \frac{C}{t_k} + \frac{C}{t_k^r} \int_{Q' \times I} \left| \left(\nabla_\alpha \phi_k \middle| \frac{1}{\delta_k} \nabla_3 \phi_k \right) \right|^{1-r} dx \\ & \leq \frac{C}{t_k} + \frac{C}{t_k^r} \|(\nabla_\alpha \phi_k | (1/\delta_k) \nabla_3 \phi_k)\|_{L^1(Q' \times I; \mathbb{R}^{3 \times 3})}^{1-r} \rightarrow 0, \end{aligned}$$

where we used the fact that, thanks to (5.39), $t_k \rightarrow +\infty$. But as W^∞ is positively 1-homogeneous, we get from (5.47) that

$$\frac{d\mu^\sigma}{d|\bar{b}^\sigma|}(x_0) = \lim_{k \rightarrow +\infty} \int_{Q' \times I} W^\infty \left(\nabla_\alpha \phi_k \middle| \frac{1}{\delta_k} \nabla_3 \phi_k \right) dx.$$

Using Remark 4.6, we can assume without loss of generality that $\phi_k = 0$ on a neighborhood of $\partial Q' \times I$. We now define

$$\tilde{\phi}_k(x_\alpha, x_3) := \phi_k(x_\alpha, x_3) + \delta_k x_3 \left(\frac{d\bar{b}}{d|\bar{b}^\sigma|}(x_0) - \frac{1}{\delta_k} \int_{Q' \times I} \nabla_3 \phi_k(y) dy \right).$$

Then, $\tilde{\phi}_k \in W^{1,1}(Q' \times I; \mathbb{R}^3)$, $\tilde{\phi}_k(\cdot, x_3)$ is Q' -periodic for \mathcal{L}^1 -a.e. $x_3 \in I$ and

$$\frac{1}{\delta_k} \int_{Q' \times I} \nabla_3 \tilde{\phi}_k dy = \frac{d\bar{b}}{d|\bar{b}^\sigma|}(x_0).$$

Hence $\tilde{\phi}_k$ is an admissible test function for $\mathcal{Q}^*(W^\infty) \left(0 \middle| \frac{d\bar{b}}{d|\bar{b}^\sigma|}(x_0) \right)$ and using the Lipschitz property (4.4), we get that

$$\frac{d\mu^\sigma}{d|\bar{b}^\sigma|}(x_0) \geq \limsup_{k \rightarrow +\infty} \int_{Q' \times I} W \left(\nabla_\alpha \tilde{\phi}_k \middle| \frac{1}{\delta_k} \nabla_3 \tilde{\phi}_k \right) dy - L \limsup_{k \rightarrow +\infty} \left| \frac{d\bar{b}}{d|\bar{b}^\sigma|}(x_0) - \frac{1}{\delta_k} \int_{Q' \times I} \nabla_3 \phi_k(y) dy \right|.$$

Finally, relation (5.46) ensures that the last term in the previous inequality is actually zero and thus, from Proposition 3.4,

$$\frac{d\mu^\sigma}{d|\bar{b}^\sigma|}(x_0) \geq \mathcal{Q}^*(W^\infty) \left(0 \middle| \frac{d\bar{b}}{d|\bar{b}^\sigma|}(x_0) \right) = (\mathcal{Q}^*W)^\infty \left(0 \middle| \frac{d\bar{b}}{d|\bar{b}^\sigma|}(x_0) \right).$$

6 The upper bound

Lemma 6.1. *For any $(u, \bar{b}) \in BV(\omega; \mathbb{R}^3) \times \mathcal{M}(\omega; \mathbb{R}^3)$, then $J_{\{\varepsilon_n\}}(u, \bar{b}, \omega) \leq E(u, \bar{b})$.*

Proof. Let $(u, \bar{b}) \in BV(\omega; \mathbb{R}^3) \times \mathcal{M}(\omega; \mathbb{R}^3)$. As in the proof of the lower bound, according to the Besicovitch Decomposition Theorem, we can decompose \bar{b} into the sum of three mutually singular measures \bar{b}^a , \bar{b}^s and \bar{b}^σ such that $\bar{b} = \bar{b}^a + \bar{b}^s + \bar{b}^\sigma$ where $\bar{b}^a \ll \mathcal{L}^2$, $\bar{b}^s \ll |D_\alpha^s u|$.

Step 1. Assume first that $\partial\omega$ is Lipschitz. Then by the locality result Lemma 4.7, it is enough to check that

$$\frac{dJ_{\{\varepsilon_n\}}(u, \bar{b}, \cdot)}{d\mathcal{L}^2}(x_0) \leq \mathcal{Q}^*W \left(\nabla_\alpha u(x_0) \middle| \frac{d\bar{b}}{d\mathcal{L}^2}(x_0) \right) \quad \text{for } \mathcal{L}^2\text{-a.e. } x_0 \in \omega, \quad (6.1)$$

$$\frac{dJ_{\{\varepsilon_n\}}(u, \bar{b}, \cdot)}{d|D_\alpha^s u|}(x_0) \leq (\mathcal{Q}^*W)^\infty \left(\frac{dD_\alpha u}{d|D_\alpha^s u|}(x_0) \middle| \frac{d\bar{b}}{d|D_\alpha^s u|}(x_0) \right) \quad \text{for } |D_\alpha^s u|\text{-a.e. } x_0 \in \omega, \quad (6.2)$$

$$\frac{dJ_{\{\varepsilon_n\}}(u, \bar{b}, \cdot)}{d|\bar{b}^\sigma|}(x_0) \leq (\mathcal{Q}^*W)^\infty \left(0 \middle| \frac{d\bar{b}}{d|\bar{b}^\sigma|}(x_0) \right) \quad \text{for } |\bar{b}^\sigma|\text{-a.e. } x_0 \in \omega. \quad (6.3)$$

Indeed, if the three previous properties hold, since $J_{\{\varepsilon_n\}}(u, \bar{b}, \cdot)$ is (the trace of) a Radon measure absolutely continuous with respect to $\mathcal{L}^2 + |D_\alpha u| + |\bar{b}|$, we deduce that

$$\begin{aligned} J_{\{\varepsilon_n\}}(u, \bar{b}, \omega) &\leq \int_\omega \mathcal{Q}^*W \left(\nabla_\alpha u \middle| \frac{d\bar{b}}{d\mathcal{L}^2} \right) dx + \int_{J_u} (\mathcal{Q}^*W)^\infty \left((u^+ - u^-) \otimes \nu_u, \frac{d\bar{b}}{d\mathcal{H}^1 \llcorner J_u} \right) d\mathcal{H}^1 \\ &\quad + \int_\omega (\mathcal{Q}^*W)^\infty \left(\frac{dD_\alpha u}{d|D_\alpha^c u|} \middle| \frac{d\bar{b}}{d|D_\alpha^c u|} \right) d|D_\alpha^c u| + \int_\omega (\mathcal{Q}^*W)^\infty \left(0 \middle| \frac{d\bar{b}}{d|\bar{b}^\sigma|} \right) d|\bar{b}^\sigma|, \end{aligned}$$

where we used the fact that $D_\alpha^s u = (u^+ - u^-) \otimes \nu_u \mathcal{H}^1 \llcorner J_u + D_\alpha^c u$ and that $(\mathcal{Q}^*W)^\infty$ is positively 1-homogeneous. We postpone the proof of the three above inequalities to the end of the section.

Step 2. Let us now explain how to remove the Lipschitz condition on $\partial\omega$. As in the proof of Lemma 4.7, for every $k \in \mathbb{N}$, it is possible to find an increasing sequence of open sets $\omega_k \subset\subset \omega_{k+1} \subset\subset \omega$ such that $\partial\omega_k$ is Lipschitz and $|\bar{b}|(\partial\omega_k) = 0$ for each $k \in \mathbb{N}$. By Step 1 and Lemma 4.5, there exists a sequence $\{u_n^k\} \subset W^{1,1}(\omega_k \times I; \mathbb{R}^3)$ such that $Tu_n^k = Tu$ on $\partial\omega_k \times I$, $u_n^k \rightarrow u$ in $L^1(\omega_k \times I; \mathbb{R}^3)$, $\frac{1}{\varepsilon_n} \int_I \nabla_3 u_n^k(\cdot, x_3) dx_3 \xrightarrow{*} \bar{b}$ in $\mathcal{M}(\omega_k; \mathbb{R}^3)$ as $n \rightarrow +\infty$ and

$$\limsup_{n \rightarrow +\infty} \int_{\omega_k \times I} W \left(\nabla_\alpha u_n^k \left| \frac{1}{\varepsilon_n} \nabla_3 u_n^k \right. \right) dx \leq E(u, \bar{b}, \omega_k) + \frac{1}{k} \leq E(u, \bar{b}, \omega) + \frac{1}{k}. \quad (6.4)$$

We now apply (a slight variant of) [8, Lemma 2.5] to get a sequence $\{v_n^k\} \subset W^{1,1}(\omega \setminus \bar{\omega}_k; \mathbb{R}^3)$ such that $v_n^k \rightarrow u$ in $L^1(\omega \setminus \bar{\omega}_k; \mathbb{R}^3)$, $Tv_n^k = Tu$ on $\partial\omega_k$ and $|D_\alpha v_n^k|(\omega \setminus \bar{\omega}_k) \rightarrow |D_\alpha u|(\omega \setminus \bar{\omega}_k)$ as $n \rightarrow +\infty$. Indeed an inspection of the proof of [8, Lemma 2.5] shows that, since we do not prescribe the value of the trace on $\partial\omega$, it is not necessary to assume $\partial\omega$ to be Lipschitz. Define $w_n^k(x_\alpha, x_3) := u_n^k(x_\alpha, x_3)\chi_{\omega_k}(x_\alpha) + v_n^k(x_\alpha)\chi_{\omega \setminus \bar{\omega}_k}(x_\alpha)$. As $Tu_n^k = Tv_n^k = Tu$ on $\partial\omega_k \times I$, the sequence $w_n^k \in W^{1,1}(\Omega; \mathbb{R}^3)$,

$$\lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} \|w_n^k - u\|_{L^1(\Omega; \mathbb{R}^3)} = 0, \quad \lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} |D_\alpha v_n^k|(\omega \setminus \bar{\omega}_k) = 0$$

and for any $\varphi \in \mathcal{C}_0(\omega; \mathbb{R}^3)$, we have

$$\lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} \int_\omega \varphi(x_\alpha) \cdot \left(\frac{1}{\varepsilon_n} \int_I \nabla_3 w_n^k(x_\alpha, x_3) dx_3 \right) dx_\alpha = \int_\omega \varphi(x_\alpha) \cdot d\bar{b}(x_\alpha).$$

Using the separability of $\mathcal{C}_0(\omega; \mathbb{R}^3)$ and a standard diagonalization procedure, we obtain the existence of a sequence $k_n \nearrow +\infty$ such that, setting $w_n := w_n^{k_n}$, then $w_n \rightarrow u$ in $L^1(\Omega; \mathbb{R}^3)$, $\frac{1}{\varepsilon_n} \int_I \nabla_3 w_n(\cdot, x_3) dx_3 \xrightarrow{*} \bar{b}$ in $\mathcal{M}(\omega; \mathbb{R}^3)$, $|D_\alpha v_n^{k_n}|(\omega \setminus \bar{\omega}_{k_n}) \rightarrow 0$ and by (6.4),

$$\limsup_{n \rightarrow +\infty} \int_{\omega_{k_n} \times I} W \left(\nabla_\alpha u_n^{k_n} \left| \frac{1}{\varepsilon_n} \nabla_3 u_n^{k_n} \right. \right) dx \leq E(u, \bar{b}, \omega). \quad (6.5)$$

Using the growth condition (H_1) together with (6.5), we get that

$$J_{\{\varepsilon_n\}}(u, \bar{b}, \omega) \leq \limsup_{n \rightarrow +\infty} \int_\Omega W \left(\nabla_\alpha w_n \left| \frac{1}{\varepsilon_n} \nabla_3 w_n \right. \right) dx \leq E(u, \bar{b}, \omega)$$

which concludes the proof of the upper bound. \square

Proof of (6.1). Fix a point $x_0 \in \omega$ such that

$$\frac{d\bar{b}}{d\mathcal{L}^2}(x_0), \quad \frac{dJ_{\{\varepsilon_n\}}(u, \bar{b}, \cdot)}{d\mathcal{L}^2}(x_0), \quad \frac{dD_\alpha u}{d\mathcal{L}^2}(x_0) = \nabla_\alpha u(x_0) \quad (6.6)$$

exist and are finite, which is also a Lebesgue point of u , $\nabla_\alpha u$ and $\frac{d\bar{b}}{d\mathcal{L}^2}$, a point of approximate differentiability for u , and such that

$$\frac{d|D_\alpha^s u|}{d\mathcal{L}^2}(x_0) = \frac{d|\bar{b} - \bar{b}^a|}{d\mathcal{L}^2}(x_0) = 0. \quad (6.7)$$

Observe that since \mathcal{L}^2 is singular with respect to $|D_\alpha^s u|$ and $|\bar{b} - \bar{b}^a|$, then \mathcal{L}^2 -a.e. $x_0 \in \omega$ satisfy all the above requirements.

Let $\{\rho_k\} \searrow 0^+$ be such that $|D_\alpha u|(\partial Q'(x_0, \rho_k)) = |\bar{b}|(\partial Q'(x_0, \rho_k)) = 0$ for each $k \in \mathbb{N}$. Let $\eta > 0$ and consider $\lambda > 0$ and $\varphi \in W^{1,1}(Q' \times I; \mathbb{R}^3)$ such that $\varphi(\cdot, x_3)$ is Q' -periodic for \mathcal{L}^1 -a.e. $x_3 \in I$, $\lambda \int_I \nabla_3 \varphi dy = \frac{d\bar{b}}{d\mathcal{L}^2}(x_0)$ and

$$\int_Q W(\nabla_\alpha u(x_0) + \nabla_\alpha \varphi | \lambda \nabla_3 \varphi) dx \leq \mathcal{Q}^* W \left(\nabla_\alpha u(x_0) \left| \frac{d\bar{b}}{d\mathcal{L}^2}(x_0) \right. \right) + \eta.$$

Then, defining $\varphi_n : \mathbb{R}^2 \times I \rightarrow \mathbb{R}^3$ by

$$\varphi_n(x_\alpha, x_3) := \lambda \varepsilon_n \varphi \left(\frac{x_\alpha}{\lambda \varepsilon_n}, x_3 \right), \quad (6.8)$$

it results that

$$\begin{cases} \varphi_n \rightarrow 0 \text{ in } L^1(Q'(x_0, \rho_k) \times I; \mathbb{R}^3), \\ \frac{1}{\varepsilon_n} \int_I \nabla_3 \varphi_n(\cdot, x_3) dx_3 \xrightarrow{*} \frac{d\bar{b}}{d\mathcal{L}^2}(x_0) \mathcal{L}^2 \text{ in } \mathcal{M}(Q'(x_0, \rho_k); \mathbb{R}^3). \end{cases} \quad (6.9)$$

Let $\{\varrho_n\}$ be a sequence of standard symmetric mollifiers chosen in such a way that

$$\lim_{n \rightarrow +\infty} \varepsilon_n \int_{Q'(x_0, \rho_k)} (|\bar{b} * \varrho_n| + |\nabla_\alpha(\bar{b} * \varrho_n)|) dx_\alpha = 0 \quad (6.10)$$

and set $v_n(x_\alpha, x_3) := (u * \varrho_n)(x_\alpha) + \varepsilon_n x_3 (\bar{b} * \varrho_n)(x_\alpha)$. Define the sequence

$$w_n(x_\alpha, x_3) := v_n(x_\alpha, x_3) + \varphi_n(x_\alpha, x_3) - \varepsilon_n x_3 \frac{d\bar{b}}{d\mathcal{L}^2}(x_0). \quad (6.11)$$

It results from (6.8), (6.9), (6.10), (6.11) and [3, Theorem 2.2] that

$$\begin{cases} w_n \rightarrow u \text{ in } L^1(Q'(x_0, \rho_k) \times I; \mathbb{R}^3), \\ \frac{1}{\varepsilon_n} \int_I \nabla_3 w_n(\cdot, x_3) dx_3 \xrightarrow{*} \bar{b} \text{ in } \mathcal{M}(Q'(x_0, \rho_k); \mathbb{R}^3). \end{cases}$$

Hence, taking $\{w_n\}$ as test function we get that

$$\begin{aligned} J_{\{\varepsilon_n\}}(u, \bar{b}, Q'(x_0, \rho_k)) &\leq \liminf_{n \rightarrow +\infty} \int_{Q'(x_0, \rho_k) \times I} W \left(\nabla_\alpha w_n \left| \frac{1}{\varepsilon_n} \nabla_3 w_n \right. \right) dx \\ &= \liminf_{n \rightarrow +\infty} \int_{Q'(x_0, \rho_k) \times I} W \left(\nabla_\alpha v_n + \nabla_\alpha \varphi_n \left| \frac{1}{\varepsilon_n} \nabla_3 v_n + \frac{1}{\varepsilon_n} \nabla_3 \varphi_n - \frac{d\bar{b}}{d\mathcal{L}^2}(x_0) \right. \right) dx \end{aligned}$$

and using the Lipschitz property (4.3) of W together with (6.8), it follows that

$$\begin{aligned} J_{\{\varepsilon_n\}}(u, \bar{b}, Q'(x_0, \rho_k)) &\leq \liminf_{n \rightarrow +\infty} \int_{Q'(x_0, \rho_k) \times I} W \left(\nabla_\alpha u(x_0) + \nabla_\alpha \varphi \left(\frac{x_\alpha}{\lambda \varepsilon_n}, x_3 \right) \left| \lambda \nabla_3 \varphi \left(\frac{x_\alpha}{\lambda \varepsilon_n}, x_3 \right) \right. \right) dx \\ &\quad + L \limsup_{n \rightarrow +\infty} \int_{Q'(x_0, \rho_k) \times I} |\nabla_\alpha v_n - \nabla_\alpha u(x_0)| dx \\ &\quad + L \limsup_{n \rightarrow +\infty} \int_{Q'(x_0, \rho_k) \times I} \left| \frac{1}{\varepsilon_n} \nabla_3 v_n - \frac{d\bar{b}}{d\mathcal{L}^2}(x_0) \right| dx. \end{aligned} \quad (6.12)$$

Observe that $\nabla_\alpha v_n(x_\alpha, x_3) = (\nabla_\alpha u * \varrho_n)(x_\alpha) + (D_\alpha^s u * \varrho_n)(x_\alpha) + \varepsilon_n x_3 \nabla_\alpha(\bar{b} * \varrho_n)(x_\alpha)$ hence,

$$\begin{aligned} \int_{Q'(x_0, \rho_k) \times I} |\nabla_\alpha v_n - \nabla_\alpha u(x_0)| dx &\leq \int_{Q'(x_0, \rho_k)} |\nabla_\alpha u * \varrho_n - \nabla_\alpha u(x_0)| dx_\alpha \\ &\quad + \int_{Q'(x_0, \rho_k)} (|D_\alpha^s u * \varrho_n| + \varepsilon_n |\nabla_\alpha(\bar{b} * \varrho_n)|) dx_\alpha. \end{aligned}$$

Thus, according to (6.10), [3, Theorem 2.2], the fact that $\nabla_\alpha u * \varrho_n \rightarrow \nabla_\alpha u$ in $L^1_{\text{loc}}(\omega; \mathbb{R}^{3 \times 2})$ and that $|D_\alpha^s u|(\partial Q'(x_0, \rho_k)) = 0$ for each $k \in \mathbb{N}$, we get that

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \int_{Q'(x_0, \rho_k) \times I} |\nabla_\alpha v_n - \nabla_\alpha u(x_0)| dx &\leq \int_{Q'(x_0, \rho_k)} |\nabla_\alpha u(x_\alpha) - \nabla_\alpha u(x_0)| dx_\alpha \\ &\quad + |D_\alpha^s u|(Q'(x_0, \rho_k)). \end{aligned} \quad (6.13)$$

Similarly, since $(1/\varepsilon_n) \nabla_3 v_n = \bar{b} * \varrho_n$, it implies that

$$\begin{aligned} \int_{Q'(x_0, \rho_k) \times I} \left| \frac{1}{\varepsilon_n} \nabla_3 v_n - \frac{d\bar{b}}{d\mathcal{L}^2}(x_0) \right| dx &\leq \int_{Q'(x_0, \rho_k)} \left| \left(\frac{d\bar{b}}{d\mathcal{L}^2} * \varrho_n \right)(x_\alpha) - \frac{d\bar{b}}{d\mathcal{L}^2}(x_0) \right| dx_\alpha \\ &\quad + \int_{Q'(x_0, \rho_k)} |(\bar{b} - \bar{b}^a) * \varrho_n|(x_\alpha) dx_\alpha. \end{aligned}$$

Since $|\bar{b} - \bar{b}^a|(\partial Q'(x_0, \rho_k)) = 0$ for each $k \in \mathbb{N}$ and $\frac{d\bar{b}}{d\mathcal{L}^2} * \varrho_n \rightarrow \frac{d\bar{b}}{d\mathcal{L}^2}$ in $L^1_{\text{loc}}(\omega; \mathbb{R}^3)$, it yields

$$\limsup_{n \rightarrow +\infty} \int_{Q'(x_0, \rho_k) \times I} \left| \frac{1}{\varepsilon_n} \nabla_3 v_n - \frac{d\bar{b}}{d\mathcal{L}^2}(x_0) \right| dx \leq \int_{Q'(x_0, \rho_k)} \left| \frac{d\bar{b}}{d\mathcal{L}^2}(x_\alpha) - \frac{d\bar{b}}{d\mathcal{L}^2}(x_0) \right| dx_\alpha + |\bar{b} - \bar{b}^a|(Q'(x_0, \rho_k)). \quad (6.14)$$

Gathering (6.12), (6.13) and (6.14) and using the Riemann-Lebesgue Lemma, we get that

$$\begin{aligned} J_{\{\varepsilon_n\}}(u, \bar{b}, Q'(x_0, \rho_k)) &\leq \rho_k^2 \mathcal{Q}^* W \left(\nabla_\alpha u(x_0) \left| \frac{d\bar{b}}{d\mathcal{L}^2}(x_0) \right. \right) + \rho_k^2 \eta \\ &\quad + L |D_\alpha^s u|(Q'(x_0, \rho_k)) + L |\bar{b} - \bar{b}^a|(Q'(x_0, \rho_k)) \\ &\quad + L \int_{Q'(x_0, \rho_k)} |\nabla_\alpha u(x_\alpha) - \nabla_\alpha u(x_0)| dx_\alpha \\ &\quad + L \int_{Q'(x_0, \rho_k)} \left| \frac{d\bar{b}}{d\mathcal{L}^2}(x_\alpha) - \frac{d\bar{b}}{d\mathcal{L}^2}(x_0) \right| dx_\alpha. \end{aligned}$$

Now dividing the previous inequality by ρ_k^2 , sending $k \rightarrow +\infty$ and exploiting properties (6.6) and (6.7) of the point x_0 , it leads to

$$\frac{dJ_{\{\varepsilon_n\}}(u, \bar{b}, \cdot)}{d\mathcal{L}^2}(x_0) \leq \mathcal{Q}^* W \left(\nabla_\alpha u(x_0) \left| \frac{d\bar{b}}{d\mathcal{L}^2}(x_0) \right. \right) + \eta$$

and the arbitrariness of η gives the desired claim.

Proof of (6.2). The proof develops in the same spirit of that in [3, Proposition 5.49] (see also [2]). Let us introduce an auxiliary function $f : \mathbb{R}^{3 \times 3} \rightarrow [0, +\infty)$ defined by

$$f(\xi) := \sup_{t > 0} \frac{\mathcal{Q}^* W(t\xi) - \mathcal{Q}^* W(0)}{t}.$$

It turns out that f is a positively 1-homogeneous continuous function. Moreover, by (4.3) there exists $L > 0$ such that

$$f(\xi) \leq L|\xi| \quad \text{and} \quad |f(\xi) - f(\xi')| \leq L|\xi - \xi'| \quad \text{for every } \xi, \xi' \in \mathbb{R}^{3 \times 3}. \quad (6.15)$$

Using the growth properties of differential quotients of convex functions, it is easily seen from Proposition 3.2 that if $z, b \in \mathbb{R}^3$ and $\nu \in \mathbb{S}^1$, then $f(z \otimes \nu |b|) = (\mathcal{Q}^* W)^\infty(z \otimes \nu |b|)$.

Fix a standard sequence of mollifiers $\{\varrho_j\}$. Then by [3, Theorem 2.2], we have that $(u * \varrho_j, \bar{b} * \varrho_j) \in W^{1,1}(\omega; \mathbb{R}^3) \times L^1(\omega; \mathbb{R}^3)$, $u * \varrho_j \rightarrow u$ in $L^1_{\text{loc}}(\omega; \mathbb{R}^3)$ and $\bar{b} * \varrho_j \xrightarrow{*} \bar{b}$ in $\mathcal{M}_{\text{loc}}(\omega; \mathbb{R}^3)$.

Using the Besicovitch Decomposition Theorem we can write $(D_\alpha u | \bar{b}) = (D_\alpha^s u | \bar{b}^s) + \lambda^s$ for some singular measure $\lambda^s \in \mathcal{M}(\omega; \mathbb{R}^{3 \times 3})$ with respect to $|D_\alpha^s u|$. Consider $x_0 \in \omega$ satisfying

$$\frac{d\lambda^s}{d|D_\alpha^s u|}(x_0) = \frac{d\mathcal{L}^2}{d|D_\alpha^s u|}(x_0) = 0, \quad (6.16)$$

such that

$$\frac{dD_\alpha^s u}{d|D_\alpha^s u|}(x_0) = \frac{dD_\alpha u}{d|D_\alpha^s u|}(x_0) \text{ is a rank one matrix,} \quad \frac{d\bar{b}^s}{d|D_\alpha^s u|}(x_0) = \frac{d\bar{b}}{d|D_\alpha^s u|}(x_0). \quad (6.17)$$

Assume further that x_0 is a Lebesgue point of

$$f \left(\frac{dD_\alpha u}{d|D_\alpha^s u|} \left| \frac{d\bar{b}}{d|D_\alpha^s u|} \right. \right) \quad (6.18)$$

with respect to $|D_\alpha^s u|$ and that

$$\frac{dJ_{\{\varepsilon_n\}}(u, \bar{b}, \cdot)}{d|D_\alpha^s u|}(x_0) \quad (6.19)$$

exists and is finite. Note that by Alberti's Rank One Theorem [1], $|D_\alpha^s u|$ almost every points $x_0 \in \omega$ satisfy these properties. Let $\{\rho_k\} \searrow 0^+$ be such that $|D_\alpha^s u|(\partial Q'(x_0, \rho_k)) = |\lambda^s|(\partial Q'(x_0, \rho_k)) = 0$ for every $k \in \mathbb{N}$.

By Remark 4.2 together with the sequential lower semicontinuity of $J_{\{\varepsilon_n\}}$, we get that

$$\begin{aligned} J_{\{\varepsilon_n\}}(u, \bar{b}, Q'(x_0, \rho_k)) &\leq \liminf_{j \rightarrow +\infty} J_{\{\varepsilon_n\}}(u * \varrho_j, \bar{b} * \varrho_j, Q'(x_0, \rho_k)) \\ &= \liminf_{j \rightarrow +\infty} \int_{Q'(x_0, \rho_k)} \mathcal{Q}^* W(\nabla_\alpha(u * \varrho_j) | \bar{b} * \varrho_j) dx_\alpha \\ &= \liminf_{j \rightarrow +\infty} \int_{Q'(x_0, \rho_k)} \mathcal{Q}^* W((D_\alpha u | \bar{b}) * \varrho_j) dx_\alpha, \end{aligned}$$

where we used the fact that $\nabla_\alpha(u * \varrho_j) = (D_\alpha u) * \varrho_j$. By definition of f , it follows that

$$J_{\{\varepsilon_n\}}(u, \bar{b}, Q'(x_0, \rho_k)) \leq \liminf_{j \rightarrow +\infty} \int_{Q'(x_0, \rho_k)} f((D_\alpha u | \bar{b}) * \varrho_j) dx_\alpha + \mathcal{Q}^* W(0) \rho_k^2$$

and using its Lipschitz property (6.15), we get that

$$\begin{aligned} J_{\{\varepsilon_n\}}(u, \bar{b}, Q'(x_0, \rho_k)) &\leq \liminf_{j \rightarrow +\infty} \int_{Q'(x_0, \rho_k)} f((D_\alpha^s u | \bar{b}^s) * \varrho_j) dx_\alpha + \mathcal{Q}^* W(0) \rho_k^2 \\ &\quad + L \limsup_{j \rightarrow +\infty} \int_{Q'(x_0, \rho_k)} |\lambda^s * \varrho_j| dx_\alpha. \end{aligned}$$

Since $|\lambda^s|(\partial Q'(x_0, \rho_k)) = 0$ for each $k \in \mathbb{N}$, then [3, Theorem 2.2] implies that

$$\begin{aligned} J_{\{\varepsilon_n\}}(u, \bar{b}, Q'(x_0, \rho_k)) &\leq \liminf_{j \rightarrow +\infty} \int_{Q'(x_0, \rho_k)} f((D_\alpha^s u | \bar{b}^s) * \varrho_j) dx_\alpha + \mathcal{Q}^* W(0) \rho_k^2 \\ &\quad + L |\lambda^s|(Q'(x_0, \rho_k)). \end{aligned}$$

As $(D_\alpha^s u | \bar{b}^s) * \varrho_j \xrightarrow{*} (D_\alpha^s u | \bar{b}^s)$ in $\mathcal{M}_{\text{loc}}(\omega; \mathbb{R}^{3 \times 3})$ as $j \rightarrow +\infty$, in particular we have that

$$(D_\alpha^s u | \bar{b}^s) * \varrho_j \xrightarrow{j \rightarrow +\infty} (D_\alpha^s u | \bar{b}^s) \text{ in } \mathcal{M}(Q'(x_0, \rho_k); \mathbb{R}^{3 \times 3}).$$

Moreover as $|D_\alpha^s u|(\partial Q'(x_0, \rho_k)) = 0$, it follows from [3, Theorem 2.2] that

$$\int_{Q'(x_0, \rho_k)} |(D_\alpha^s u | \bar{b}^s) * \varrho_j| dx_\alpha \xrightarrow{j \rightarrow +\infty} |(D_\alpha^s u | \bar{b}^s)|(Q'(x_0, \rho_k)).$$

Hence, applying Reshetnyak's Continuity Theorem (see *e.g.* [3, Theorem 2.39]), we infer that

$$\begin{aligned} J_{\{\varepsilon_n\}}(u, \bar{b}, Q'(x_0, \rho_k)) &\leq \int_{Q'(x_0, \rho_k)} f\left(\frac{dD_\alpha^s u}{d|D_\alpha^s u|} \middle| \frac{d\bar{b}^s}{d|D_\alpha^s u|}\right) d|D_\alpha^s u| + \mathcal{Q}^* W(0) \rho_k^2 \\ &\quad + L |\lambda^s|(Q'(x_0, \rho_k)), \end{aligned}$$

where we used the fact that f is positively 1-homogeneous and that $(D_\alpha^s u | \bar{b}^s)$ is absolutely continuous with respect to $|D_\alpha^s u|$. Dividing the previous inequality by $|D_\alpha^s u|(Q'(x_0, \rho_k))$, sending $k \rightarrow +\infty$ and using (6.16), (6.17), (6.18) and (6.19), we deduce that

$$\frac{dJ_{\{\varepsilon_n\}}(u, \bar{b}, \cdot)}{d|D_\alpha^s u|}(x_0) \leq f\left(\frac{dD_\alpha u}{d|D_\alpha^s u|}(x_0) \middle| \frac{d\bar{b}}{d|D_\alpha^s u|}(x_0)\right) = (\mathcal{Q}^* W)^\infty\left(\frac{dD_\alpha u}{d|D_\alpha^s u|}(x_0) \middle| \frac{d\bar{b}}{d|D_\alpha^s u|}(x_0)\right)$$

since $\frac{dD_\alpha u}{d|D_\alpha^s u|}(x_0)$ is a rank one matrix.

Proof of (6.3). The proof for estimating from above the term concerning the singular part is analogous to the previous one.

Using the Besicovitch Decomposition Theorem we can write $(D_\alpha u|\bar{b}) = (0|\bar{b}^\sigma) + \lambda^\sigma$ for some singular measure $\lambda^\sigma \in \mathcal{M}(\omega; \mathbb{R}^{3 \times 3})$ with respect to $|\bar{b}^\sigma|$. Consider $x_0 \in \omega$ to be a Lebesgue point of

$$f \left(0 \left| \frac{d\bar{b}^\sigma}{d|\bar{b}^\sigma|} \right| \right) \quad (6.20)$$

with respect to $|\bar{b}^\sigma|$ satisfying

$$\frac{d|\bar{b} - \bar{b}^\sigma|}{d|\bar{b}^\sigma|}(x_0) = \frac{d\lambda^\sigma}{d|\bar{b}^\sigma|}(x_0) = \frac{d\mathcal{L}^2}{d|\bar{b}^\sigma|}(x_0) = 0, \quad (6.21)$$

and such that

$$\frac{dJ_{\{\varepsilon_n\}}(u, \bar{b}, \cdot)}{d|\bar{b}^\sigma|}(x_0) \quad (6.22)$$

exists and is finite. Note that $|\bar{b}^\sigma|$ almost every points $x_0 \in \omega$ satisfy these properties. Let $\{\rho_k\} \searrow 0^+$ be such that $|\bar{b}^\sigma|(\partial Q'(x_0, \rho_k)) = |\lambda^\sigma|(\partial Q'(x_0, \rho_k)) = 0$ for every $k \in \mathbb{N}$.

Arguing exactly as in the previous subsection, we obtain that

$$\begin{aligned} J_{\{\varepsilon_n\}}(u, \bar{b}, Q'(x_0, \rho_k)) &\leq \int_{Q'(x_0, \rho_k)} f \left(0 \left| \frac{d\bar{b}^\sigma}{d|\bar{b}^\sigma|} \right| \right) d|\bar{b}^\sigma| + \mathcal{Q}^* W(0) \rho_k^2 \\ &\quad + L |\lambda^\sigma|(Q'(x_0, \rho_k)). \end{aligned}$$

Dividing the previous inequality by $|\bar{b}^\sigma|(Q'(x_0, \rho_k))$, sending $k \rightarrow +\infty$ and using (6.20), (6.21) and (6.22), it implies that

$$\frac{dJ_{\{\varepsilon_n\}}(u, \bar{b}, \cdot)}{d|\bar{b}^\sigma|}(x_0) \leq f \left(0 \left| \frac{d\bar{b}}{d|\bar{b}^\sigma|} \right| (x_0) \right) = (\mathcal{Q}^* W)^\infty \left(0 \left| \frac{d\bar{b}}{d|\bar{b}^\sigma|} \right| (x_0) \right).$$

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